MATHEMATICA MORAVICA
VOL. 2 (1998), 91–96

ON THE LOCATION OF THE ZEROS OF POLYNOMIALS

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Summary. In this paper we determine in the complex plane regions containing the zeros of the polynomial

$$(1)\quad P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n, \qquad n \geq 3.$$

We also obtain an expression which represents an upper bound for the moduli of the zeros of $P(z)$.

1. The location of the zeros of the polynomial (1) in the complex plane, depending on its coefficients a_k, $k = 1, 2, \cdots, n$, was investigated by many authors (see e.g. [1] and [2]). Here we quote a result due to P. Montel [3] which is as follows:
(R_1): *All the zeros of the polynomial (1) lie in the region*

$$(2) \qquad |z| < 2M,$$

where

$$(3) \qquad M = \max |a_k|^{\frac{1}{k}}, \qquad k = 1, 2, \cdots, n.$$

Representing the polynomial (1) in the form

$$(4) \qquad P(z) = z^{n-1}(z - a_1) + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n,$$

M. Tomić [4] considered the zeros of the polynomial (1) in the halfplanes $|z + a_1| \geq |z - 1|$, $|z + a_1| < |z - 1|$ and the halfplanes $|z + a_1| \geq |z|$, $|z + a_1| < |z|$ of the complex plane and obtained interesting results.

AMS Subject Classification 1991. Primary: 12D10.

Key words and phrases: regions of the zeros, upper bound for the moduli of the zeros, the zeros in the halfplanes.

2. In this paper, the zeros of the polynomial (1) are considered in certain circular regions and also in the halfplanes $|z+a_1| \geq |z|$, $|z+a_1| < |z|$ of the complex plane and the following theorem is proved.

Theorem. *For fixed positive parameter s, let*

$$M_s = \max |a_k|^{\frac{1}{s+k-2}}, \quad k = 2, 3, \cdots, n. \tag{5}$$

Then all the zeros of the polynomial (1) lie in the union of the regions

$$|z + a_1| < B, \qquad |z| < M_s + (M_s^s) B^{-1}, \tag{6}$$

where B is an arbitrary positive constant.

Proof: From(5) we have

$$|a_2| \leq M_s^s, \qquad |a_3| \leq M_s^{s+1}, \cdots, |a_n| \leq M_s^{s+n-2}. \tag{7}$$

Taking into account (7), from (4) for $|z| > M_s$ we have

$$|P(z)| \geq |z|^{n-1} \left\{ |z + a_1| - |z|^{s-1} \left[\frac{|a_2|}{|z|^s} + \frac{|a_3|}{|z|^{s+1}} + \cdots + \frac{|a_n|}{|z|^{s+n-2}} \right] \right\} >$$

$$> |z|^{n-1} \left\{ |z + a_1| - |z|^{s-1} \left[\left(\frac{M_s}{|z|} \right)^s + \left(\frac{M_s}{|z|} \right)^{s+1} + \cdots + \left(\frac{M_s}{|z|} \right)^{s+n-2} + \cdots \right] \right\} =$$

$$= |z|^{n-1} \left\{ |z + a_1| - \frac{M_s^s}{|z| - M_s} \right\},$$

that is

$$|P(z)| > |z|^{n-1} \left\{ |z + a_1| - \frac{M_s^s}{|z| - M_s} \right\}. \tag{8}$$

For $|z + a_1| \geq B$, that is, for points in the complex plane which do not lie in the region $|z + a_1| < B$, we obtain from (8)

$$|P(z)| > |z|^{n-1} \left\{ B - \frac{M_s^s}{|z| - M_s} \right\}. \tag{9}$$

According to (9), for $|z| > M_s$ in the region $|z+a_1| \geq B$ we have $|P(z)| > 0$ for $|z| \geq M_s + (M_s^s)B^{-1}$. This means that $|P(z)| \neq 0$ at points of the complex plane which do not lie in the regions $|z + a_1| < B$ and $|z| < M_s + (M_s^s)B^{-1}$. From this we deduce that all the zeros of the polynomial (1) must lie in the union of the circular regions (6), which finishes the proof of Theorem.

3. Taking for B and s different positive values we can obtain from Theorem several particular results. Here we list some particular cases.

3.1. For $B = \frac{1}{2}\left[M_s - |a_1| + \sqrt{(M_s - |a_1|)^2 + 4M_s^s}\right]$ (when $M_s + (M_s^s)B^{-1} = |a_1| + B$, i.e. the case when the circle $|z + a_1| = B$ is contained in the circle $|z| = M_s + (M_s^s)B^{-1}$ and touches it):

(R_{31}): *All the zeros of the polynomial (1) lie in the region*

$$|z| < \frac{1}{2}\left[M_s + |a_1| + \sqrt{(M_s - |a_1|)^2 + 4M_s^s}\right]. \tag{10}$$

3.2. For $B = \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right]$ (when $B = M_s + (M_s^s)B^{-1}$):

(R_{32}): *All the zeros of the polynomial* (1) *lie in the union of the regions*

$$\begin{aligned} |z + a_1| &< \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right], \\ |z| &< \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right]. \end{aligned} \tag{11}$$

It is not difficult to see that

$$\begin{aligned} &\frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right] \le \\ &\le \frac{1}{2}\left[M_s + |a_1| + \sqrt{(M_s - |a_1|)^2 + 4M_s^s}\right], \end{aligned} \tag{12}$$

where in (12) we have the equality sign when $a_1 = 0$.

If $a_1 = 0$, then according to (11) it follows that all the zeros of the polynomial (1) lie in the region

$$|z| < \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right]. \tag{13}$$

The circular regions (11) are symmetric with respect to the line $|z + a_1| = |z|$, for $a_1 \neq 0$. Taking this into account, the result (R_{32}) can be given in the form:

(R'_{32}): *In the halfplane*

$$|z + a_1| \geq |z| \tag{14}$$

the zeros of the polynomial (1) lie in the region

$$|z| < \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right]. \tag{15}$$

In the halfplane

$$|z + a_1| < |z| \tag{16}$$

the zeros of the polynomial (1) lie in the region

$$|z + a_1| < \frac{1}{2}\left[M_s + \sqrt{M_s^2 + 4M_s^s}\right]. \tag{17}$$

Let us consider the halifplane (14). On account of (12), the region (15) is contained in the circular region (10).

If we consider the complex plane as the union of the halfplanes (14) and (16), then according to (R_{31}) and (R'_{32}) we can conclude that:

(R''_3): *In the halfplane (14) the zeros of the polynomial (1) lie in the region (15), but in the halfplane (16) the zeros of the polynomial (1) lie in the region (10).*

4. For $s = 2$, from (5) we obtain

$$M_2 = \max |a_k|^{\frac{1}{k}}, \qquad k = 2, 3, \ldots, n. \tag{18}$$

In this case the results (R_{31}) and (R'_{32}) reduce, respectively, to:

(R_{41}): *All the zeros of the polynomial* (1) *lie in the region*

$$|z| < \frac{1}{2}\left[M_2 + |a_1| + \sqrt{(M_2 - |a_1|)^2 + 4M_2^2}\right]. \tag{19}$$

(R'_{42}): *In the halfplane (14) the zeros of the polynomial (1)* lie in the region

$$|z| < \frac{1}{2}\left(1 + \sqrt{5}\right) M_2. \tag{20}$$

In the halfplane (16) the zeros of the polynomial (1) lie in the region

$$|z+a_1| < \frac{1}{2}\left(1+\sqrt{5}\right) M_2. \tag{21}$$

Because of $M_2 \leq M$ and $|a_1| \leq M$, we obtain

$$\frac{1}{2}\left[M_2 + |a_1| + \sqrt{(M_2-|a_1|)^2 + 4M_2^2}\right] \leq 2M, \tag{22}$$

where in (22) we have the equality sign vhen $M_2 = |a_1| = M$.

We also obtain

$$\frac{1}{2}\left(1+\sqrt{5}\right) M_2 \leq \frac{1}{2}\left(1+\sqrt{5}\right) M < 2M. \tag{23}$$

Taking into account (22) we conclude that the region (19) is contained in Montel's region (2), except when $M_2 = M = |a_1|$ in which case the two regions coincide.

In view of (R_{41}) and (R'_{42}) we can also conclude that:

(R''_4): *In the halfplane (14) the zeros of the polynomial (1) lie in the region* (20), *but in the halfplane (16) the zeros of the polynomial (1) lie in the region* (19).

Both the regions (19) and (20) are contained in Montel's region (2).

5. The case when $s \to \infty$. From (5) we have

$$\lim_{s\to\infty} M_s = 1; \quad \lim_{s\to\infty} M_s^s = A_2 = \max|a_k|, \quad k = 2,3,\cdots,n.$$

In this case the results (R_{31}), (R_{32}) and (R'_{32}) reduce to the results given in [5].

6. References

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Received November 21, 1997.