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Inequalities of Jensen Type for h-Convex Functions on Linear Spaces

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Abstract. Some inequalities of Jensen type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. Introduction

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([38]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [28, 29, 31, 44, 47, 48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([31]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

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$$(1.3)\qquad f(tx+(1-t)y)\leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [31] and [45] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3 ([7]). *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx+(1-t)y)\leq t^s f(x)+(1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1,2,7,8,26,27,39,41,50].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx+(1-t)y) &= \|tx+(1-t)y\|^s \leq (t\|x\|+(1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([53]). *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$(1.4)\qquad f(tx+(1-t)y)\leq h(t) f(x)+h(1-t) f(y)$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [6, 42, 49, 51–53].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1–4, 6, 9–37, 40–42, 45–52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t) h(s) \text{ for any } t, s \in J. \tag{1.6}$$

If the inequality (1.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.6) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x + c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

The case of h-convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable. However, with similar proofs they can be extended to h-convex function defined on convex subsets in linear spaces.

Theorem 1. *Let $h : J \to [0, \infty)$ be a supermultiplicative function on J. If the function $f : C \subseteq X \to [0, \infty)$ is h-convex on the convex subset C of the linear space X, then for any $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) f(x_i). \tag{1.7}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq h\left(\frac{1}{n}\right) \sum_{i=1}^{n} f(x_i). \tag{1.8}$$

Corollary 1 ([27]). *If the function $f : C \subseteq X \to [0, \infty)$ is Breckner s-convex on the convex subset C of the linear space X with $s \in (0,1)$, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n^s}\sum_{i=1}^{n} w_i^s f(x_i). \tag{1.9}$$

If $(X, \|\cdot\|)$ is a normed linear space, then for $s \in (0,1)$, $x_i \in X$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have the norm inequality

$$\left\|\sum_{i=1}^{n} w_i x_i\right\|^s \leq \sum_{i=1}^{n} w_i^s \|x_i\|^s. \tag{1.10}$$

Corollary 2. *If the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, on the convex subset C of the linear space X, then for any $x_i \in C$, $w_i > 0$, $i \in \{1, ..., n\}$, $n \geq 2$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq W_n^s \sum_{i=1}^{n} \frac{1}{w_i^s} f(x_i). \tag{1.11}$$

This result generalizes the Jensen type inequality obtained in [44] for $s = 1$.

Let K be a finite non-empty set of positive integers. We can define the index set function, see also [53]

$$J(K) := \sum_{i \in K} h(w_i) f(x_i) - h(W_K) f\left(\frac{1}{W_K}\sum_{i \in K} w_i x_i\right), \tag{1.12}$$

where $W_K := \sum_{i \in K} w_i > 0$, $x_i \in C$, $i \in K$.

We notice that if $h : [0, \infty) \to [0, \infty)$ is a supermultiplicative function on $[0, \infty)$ and the function $f : C \subseteq X \to [0, \infty)$ is h-convex on the convex subset C of the linear space X, then

$$J(K) \geq h(W_K)\left[\sum_{i \in K} h\left(\frac{w_i}{W_K}\right) f(x_i) - f\left(\frac{1}{W_K}\sum_{i \in K} w_i x_i\right)\right] \geq 0. \tag{1.13}$$

Theorem 2. *Assume that $h : [0, \infty) \to [0, \infty)$ is a supermultiplicative function on $[0, \infty)$ and the function $f : C \subseteq X \to [0, \infty)$ is h-convex on the convex subset C of the linear space X. Let M and K be finite non-empty sets of positive integers, $w_i > 0$, $x_i \in C$, $i \in K \cup M$. Then*

$$J(K \cup M) \geq J(K) + J(M) \geq 0, \tag{1.14}$$

i.e., J is a superadditive index set functional.

This results was proved in an equivalent form in [53] for functions of a real variable. The proof is similar for functions defined on convex sets in linear spaces.

Corollary 3. *With the assumptions of Theorem 2 and if we note* $M_k := \{1, ..., k\}$, *then*

$$J(M_n) \geq J(M_{n-1}) \geq ... \geq J(M_2) \geq 0 \tag{1.15}$$

and

$$\begin{aligned} J(M_n) \geq \max_{1\leq i<j\leq n} \Big\{ & h(w_i) f(x_i) + h(w_j) f(x_j) \\ & - h(w_i + w_j) f\left(\frac{w_i x_i + w_j x_j}{w_i + w_j}\right) \Big\} \\ \geq 0. & \end{aligned} \tag{1.16}$$

If we consider the functional

$$J_s(K) := \sum_{i\in K} w_i^s \|x_i\|^s - \left\| \sum_{i\in K} w_i x_i \right\|^s$$

for $s \in (0,1)$, then we have the norm inequalities

$$\begin{aligned} \sum_{i=1}^{n} w_i^s \|x_i\|^s - \left\| \sum_{i=1}^{n} w_i x_i \right\|^s & \geq \sum_{i=1}^{n-1} w_i^s \|x_i\|^s - \left\| \sum_{i=1}^{n-1} w_i x_i \right\|^s \\ & \geq ... \geq \sum_{i=1}^{2} w_i^s \|x_i\|^s - \left\| \sum_{i=1}^{2} w_i x_i \right\|^s \geq 0 \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} & \sum_{i=1}^{n} w_i^s \|x_i\|^s - \left\| \sum_{i=1}^{n} w_i x_i \right\|^s \\ & \geq \max_{1\leq i<j\leq n} \left\{ w_i^s \|x_i\|^s + w_j^s \|x_j\|^s - \|w_i x_i + w_j x_j\|^s \right\} \geq 0 \end{aligned} \tag{1.18}$$

where $w_i \geq 0$, $x_i \in X$, $i \in \{1, ..., n\}$, $n \geq 2$.

2. More Jensen Type Results

Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$
\begin{aligned}
h(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \qquad z \in D(0,1); \\
h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \qquad z \in \mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \qquad z \in \mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad z \in D(0,1).
\end{aligned} \tag{2.1}
$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$
\begin{aligned}
h(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z), \qquad z \in \mathbb{C}, \\
h(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right), \qquad z \in D(0,1); \\
h(z) &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1} = \sin^{-1}(z), \qquad z \in D(0,1); \\
h(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \qquad z \in D(0,1) \\
h(z) &= {}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0, \\
&\quad z \in D(0,1);
\end{aligned} \tag{2.2}
$$

where Γ is *Gamma function.*

The following result may provide many examples of supemultiplicative functions.

Theorem 3. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0,1] \to [0,\infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. Then h_r is supemultiplicative on $[0,1]$.*

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$:

$$
\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \tag{2.3}
$$

for any $n \in \mathbb{N}$.

Let $t, s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{2.4}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i, \ \sum_{i=0}^{\infty} a_i (rts)^i, \quad \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (2.4) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs)$$

i.e.

$$h_r(ts) \geq h_r(t)\, h_r(s).$$

This inequality is also obviously satisfied at the end points of the interval $[0,1]$ and the proof is completed. ∎

Remark 1. *Utilising the above theorem, we then conclude that the functions*

$$h_r : [0,1] \to [0,\infty), \ h_r(t) := \frac{1-r}{1-rt}, \ r \in (0,1)$$

and

$$h_r : [0,1] \to [0,\infty), \ h_r(t) := \exp[-r(1-t)], \ r > 0$$

are supermultiplicative.

We say that the function $f : C \subseteq X \to [0,\infty)$ is r-resolvent convex with r fixed in $(0,1)$, if f is h-convex with $h(t) = \frac{1-r}{1-rt}$, i.e.

$$f(tx + (1-t)y) \leq (1-r)\left[\frac{1}{1-rt} f(x) + \frac{1}{1-r+rt} f(y)\right] \tag{2.5}$$

for any $x, y \in C$ and $t \in [0,1]$.

In particular, for $r = \frac{1}{2}$ we have $\frac{1}{2}$-resolvent convex functions defined by the condition

$$f(tx + (1-t)y) \leq \frac{1}{2-t} f(x) + \frac{1}{1+t} f(y) \tag{2.6}$$

for any $t \in [0,1]$ and $x, y \in C$.

Since

$$t < \frac{1}{2-t} < \frac{1}{t} \text{ and } 1-t < \frac{1}{1+t} < \frac{1}{1-t} \text{ for } t \in (0,1)$$

it follows that any nonnegative convex function is $\frac{1}{2}$-resolvent convex which, in its turn, is of Godunova-Levin type.

We say that the function $f : C \subseteq X \to [0, \infty)$ is r-exponential convex with r fixed in $(0, \infty)$, if f is h-convex with $h(t) = \exp[-r(1-t)]$, i.e.

$$f(tx + (1-t)y) \leq \exp[-r(1-t)] f(x) + \exp(-rt) f(y) \tag{2.7}$$

for any $t \in [0, 1]$ and $x, y \in C$.

Since

$$t \leq \exp[-r(1-t)] \text{ and } 1 - t \leq \exp(-rt) \text{ for } t \in [0, 1]$$

it follows that any nonnegative convex function is r-exponential convex with $r \in (0, \infty)$.

Corollary 4. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0, 1] \to [0, \infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. If the function $f : C \subseteq X \to [0, \infty)$ is h_r-convex on the convex subset C of the linear space X, namely*

$$f(tx + (1-t)y) \leq \frac{1}{h(r)} [h(rt) f(x) + h(r(1-t)) f(y)] \tag{2.8}$$

for any $t \in [0, 1]$ and $x, y \in C$, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{h(r)} \sum_{i=1}^{n} h\left(r \frac{w_i}{W_n}\right) f(x_i). \tag{2.9}$$

Remark 2. *If the function $f : C \subseteq X \to [0, \infty)$ is $\frac{1}{2}$-resolvent convex on C, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \leq W_n \sum_{i=1}^{n} \frac{1}{2W_n - w_i} f(x_i).$$

If the function $f : C \subseteq X \to [0, \infty)$ is r-exponential convex with r fixed in $(0, \infty)$, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} \exp\left[-r\left(1 - \frac{w_i}{W_n}\right)\right] f(x_i).$$

3. Some Related Functionals

Let us fix $K \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in K)$. Now consider the functional $J_K : S_+(K) \to \mathbb{R}$ given by

$$J_K(\mathbf{p}) := h(P_K) f\left(\frac{1}{P_K} \sum_{i \in K} p_i x_i\right) \geq 0 \tag{3.1}$$

where $S_+(K) := \left\{ \mathbf{p} = (p_i)_{i \in I} \middle| p_i \geq 0, \ i \in K \ \text{ and } \ P_K > 0 \right\}$ with $h : (0, \infty) \to (0, \infty)$ and f is nonnegative on C.

Theorem 4. *Let $h : (0, \infty) \to (0, \infty)$ be a supermultiplicative (submultiplicative) function on J. If the function $f : C \subseteq X \to [0, \infty)$ is h-convex (h-concave) on the convex subset C of the linear space X, then for any $\mathbf{p}, \mathbf{q} \in S_+(K)$ we have*

$$J_K(\mathbf{p} + \mathbf{q}) \leq (\geq) J_K(\mathbf{p}) + J_K(\mathbf{q}), \tag{3.2}$$

i.e., J_K is a subadditive (superadditive) functional on $S_+(K)$.

Proof. If the function $f : C \subseteq X \to [0, \infty)$ is h -convex, then we have for any $\mathbf{p}, \mathbf{q} \in S_+(K)$

$$\begin{aligned}
&J_K(\mathbf{p} + \mathbf{q}) \\
&= h(P_K + Q_K) f\left(\frac{1}{P_K + P_K} \sum_{i \in K} (p_i + q_i) x_i \right) \\
&= h(P_K + Q_K) f\left(\frac{P_K \cdot \frac{1}{P_K} \sum_{i \in K} p_i x_i + Q_K \cdot \frac{1}{Q_K} \sum_{i \in K} q_i x_i}{P_K + P_K} \right) \\
&\leq h(P_K + Q_K) \left[h\left(\frac{P_K}{P_K + P_K} \right) f\left(\frac{1}{P_K} \sum_{i \in K} p_i x_i \right) \right. \\
&\left. + h\left(\frac{Q_K}{P_K + P_K} \right) f\left(\frac{1}{Q_K} \sum_{i \in K} q_i x_i \right) \right] \\
&:= A.
\end{aligned} \tag{3.3}$$

Since h is supermultiplicative, then

$$h(P_K + Q_K) h\left(\frac{P_K}{P_K + P_K} \right) \leq h(P_K)$$

and

$$h(P_K + Q_K) h\left(\frac{Q_K}{P_K + P_K} \right) \leq h(Q_K)$$

which imply that

$$\begin{aligned}
A &\leq h(P_K) f\left(\frac{1}{P_K} \sum_{i \in K} p_i x_i \right) + h(Q_K) f\left(\frac{1}{Q_K} \sum_{i \in K} q_i x_i \right) \\
&= J_K(\mathbf{p}) + J_K(\mathbf{q}).
\end{aligned} \tag{3.4}$$

Making use of (3.3) and (3.4) we deduce the desired result (3.2).

The case when h is submultiplicative and $f : C \subseteq X \to [0, \infty)$ is h-concave goes likewise and the details are omitted. ∎

Corollary 5. *Let $h:(0,\infty)\to(0,\infty)$ be a submultiplicative function on J. If the function $f:C\subseteq X\to[0,\infty)$ is h-concave on the convex subset C of the linear space X, then for any $\mathbf{p},\mathbf{q}\in S_+(K)$ with $\mathbf{p}\geq\mathbf{q}$, i.e. $p_i\geq q_i$ for any $i\in K$, we have*

$$J_K(\mathbf{p})\geq J_K(\mathbf{q})\geq 0, \tag{3.5}$$

i.e., J_K is monotonic nondecreasing on $S_+(K)$.

The proof is obvious from (3.2) on noticing that

$$J_K(\mathbf{p})=J_K(\mathbf{p}-\mathbf{q}+\mathbf{q})\geq J_K(\mathbf{p}-\mathbf{q})+J_K(\mathbf{q})\geq J_K(\mathbf{q}).$$

We also have:

Corollary 6. *Let $h:(0,\infty)\to(0,\infty)$ be a submultiplicative function on J. If the function $f:C\subseteq X\to[0,\infty)$ is h-concave on the convex subset C of the linear space X, then for any $\mathbf{p},\mathbf{q}\in S_+(K)$ with $M\mathbf{p}\geq\mathbf{q}\geq m\mathbf{p}$, for some $M>m>0$, we have*

$$\frac{h(MP_K)}{h(P_K)}J_K(\mathbf{p})\geq J_K(\mathbf{q})\geq\frac{h(mP_K)}{h(P_K)}J_K(\mathbf{p}). \tag{3.6}$$

Proof. From the inequality (3.5) we have

$$J_K(M\mathbf{p})\geq J_K(\mathbf{q}).$$

However

$$\begin{aligned}J_K(M\mathbf{p})&=h(MP_K)f\left(\frac{1}{MP_K}\sum_{i\in K}Mp_ix_i\right)\\&=h(MP_K)f\left(\frac{1}{P_K}\sum_{i\in K}p_ix_i\right)=\frac{h(MP_K)}{h(P_K)}J_K(\mathbf{p}),\end{aligned}$$

which proves the first inequality in (3.6).

The second inequality can be proved similarly and the details are omitted. ∎

Further, consider the functional $L_K:S_+(K)\to\mathbb{R}$ given by

$$L_K(\mathbf{p}):=h(P_K)\sum_{i\in K}h\left(\frac{p_i}{P_K}\right)f(x_i)\geq 0, \tag{3.7}$$

where $S_+(K):=\left\{\mathbf{p}=(p_i)_{i\in I}\,\middle|\,p_i\geq 0,\ i\in K\ \text{ and }\ P_K>0\right\}$ with $h:(0,\infty)\to(0,\infty)$ and f is nonnegative on C.

Theorem 5. *Let $h:(0,\infty)\to(0,\infty)$ and $f:C\subseteq X\to[0,\infty)$. If h is convex (concave) on $(0,\infty)$ and $g:(0,\infty)\to(0,\infty)$ defined by $g(t)=\frac{h(t)}{t}$ is decreasing (increasing), then for any $\mathbf{p},\mathbf{q}\in S_+(K)$ we have*

$$L_K(\mathbf{p}+\mathbf{q})\leq(\geq)\,L_K(\mathbf{p})+L_K(\mathbf{q}). \tag{3.8}$$

Proof. If h is convex on $(0,\infty)$, then we have for any $\mathbf{p},\mathbf{q}\in S_+(K)$

$$\begin{aligned}
L_K(\mathbf{p}+\mathbf{q}) &= h(P_K+Q_K)\sum_{i\in K} h\left(\frac{p_i+q_i}{P_K+Q_K}\right) f(x_i) \\
&= h(P_K+Q_K)\sum_{i\in K} h\left(\frac{P_K\frac{p_i}{P_K}+Q_K\frac{q_i}{Q_K}}{P_K+Q_K}\right) f(x_i) \\
&\le h(P_K+Q_K)\times\sum_{i\in K}\left[\frac{P_K}{P_K+Q_K}h\left(\frac{p_i}{P_K}\right)\right. \\
&\qquad\qquad \left.+\frac{Q_K}{P_K+Q_K}h\left(\frac{q_i}{Q_K}\right)\right] f(x_i) \\
&= \frac{h(P_K+Q_K)P_K}{P_K+Q_K}\sum_{i\in K} h\left(\frac{p_i}{P_K}\right) f(x_i) \\
&\quad +\frac{h(P_K+Q_K)Q_K}{P_K+Q_K}\sum_{i\in K} h\left(\frac{q_i}{Q_K}\right) f(x_i) \\
&:= B.
\end{aligned} \tag{3.9}$$

Since $g(t)=\frac{h(t)}{t}$ is decreasing, then

$$\frac{h(P_K+Q_K)}{P_K+Q_K}\le\frac{h(P_K)}{P_K}$$

and

$$\frac{h(P_K+Q_K)}{P_K+Q_K}\le\frac{h(Q_K)}{Q_K}.$$

Therefore

$$\begin{aligned}
B &\le h(P_K)\sum_{i\in K} h\left(\frac{p_i}{P_K}\right) f(x_i)+h(Q_K)\sum_{i\in K} h\left(\frac{q_i}{Q_K}\right) f(x_i) \\
&= L_K(\mathbf{p})+L_K(\mathbf{q}).
\end{aligned} \tag{3.10}$$

Making use of (3.9) and (3.10) we deduce the desired result (3.8).

The case when h is concave and g is increasing goes likewise and the details are omitted. ∎

Corollary 7. *Let $h:(0,\infty)\to(0,\infty)$ and $f:C\subseteq X\to[0,\infty)$. If h is concave on $(0,\infty)$ and $g:(0,\infty)\to(0,\infty)$ defined by $g(t)=\frac{h(t)}{t}$ is increasing, then for any $\mathbf{p},\mathbf{q}\in S_+(K)$ with $\mathbf{p}\ge\mathbf{q}$ we have*

$$L_K(\mathbf{p})\ge L_K(\mathbf{q})\ge 0. \tag{3.11}$$

Also, for any $\mathbf{p},\mathbf{q}\in S_+(K)$ with $M\mathbf{p}\ge\mathbf{q}\ge m\mathbf{p}$, for some $M>m>0$, we have

$$\frac{h(MP_K)}{h(P_K)}L_K(\mathbf{p})\ge L_K(\mathbf{q})\ge\frac{h(mP_K)}{h(P_K)}L_K(\mathbf{p}). \tag{3.12}$$

We define the difference functional

$$\begin{aligned} S_K(\mathbf{p}) &:= L_K(\mathbf{p}) - J_K(\mathbf{p}) \\ &= h(P_K)\left[\sum_{i\in K} h\left(\frac{p_i}{P_K}\right) f(x_i) - f\left(\frac{1}{P_K}\sum_{i\in K} p_i x_i\right)\right]. \end{aligned}$$

We observe that, if h is supermultiplicative and $f : C \subseteq X \to [0,\infty)$ is h-convex, then by Jensen's type inequality (1.7) we have

$$S_K(\mathbf{p}) \geq 0 \text{ for any } \mathbf{p} \in S_+(K).$$

Proposition 1. *Let $h : (0,\infty) \to (0,\infty)$ be supermultiplicative and $f : C \subseteq X \to [0,\infty)$ a h-convex function on C. If h is concave on $(0,\infty)$ and $g : (0,\infty) \to (0,\infty)$ defined by $g(t) = \frac{h(t)}{t}$ is increasing, then for any $\mathbf{p},\mathbf{q} \in S_+(K)$*

$$S_K(\mathbf{p}+\mathbf{q}) \geq S_K(\mathbf{p}) + S_K(\mathbf{q}) \geq 0. \tag{3.13}$$

If $\mathbf{p},\mathbf{q} \in S_+(K)$ with $\mathbf{p} \geq \mathbf{q}$, then we have

$$S_K(\mathbf{p}) \geq S_K(\mathbf{q}) \geq 0. \tag{3.14}$$

Also, for any $\mathbf{p},\mathbf{q} \in S_+(K)$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, for some $M > m > 0$, we have

$$\frac{h(MP_K)}{h(P_K)} S_K(\mathbf{p}) \geq S_K(\mathbf{q}) \geq \frac{h(mP_K)}{h(P_K)} S_K(\mathbf{p}). \tag{3.15}$$

The proof follows by Theorem 4 and Theorem 5 and we omit the details.

If we take $h(t) = t$, i.e. in the case of convex functions we obtain from Proposition 1 the superadditivity and monotonicity properties of the functional

$$Je_K(\mathbf{p}) := \sum_{i\in K} p_i f(x_i) - P_K f\left(\frac{1}{P_K}\sum_{i\in K} p_i x_i\right)$$

established in ([32]).

From (3.15) we get

$$MJe_K(\mathbf{p}) \geq Je_K(\mathbf{q}) \geq mJe_K(\mathbf{p}) \tag{3.16}$$

that has been obtained in [24].

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