

Quadruple Coincidence and Common Quadruple Fixed Point for Hybrid Pair of Mappings Under New Contractive Condition

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ABSTRACT. We establish a quadruple coincidence and common quadruple fixed point theorem for hybrid pair of mappings satisfying new contractive condition. It is to be noted that to find quadruple coincidence points, we do not use the condition of continuity of any mapping involved. An example supporting to our result has also been cited. We improve, extend and generalize several known results.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X . Let $D(x, A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d , that is,

$$D(x, A) = \inf_{a \in A} d(x, a) \quad \text{and} \\ H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}, \quad \text{for all } A, B \in CB(X).$$

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [13]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. The theory of multivalued mappings has applications in control theory, convex optimization, differential inclusion and economics. For details, see [3, 4, 10, 11, 16] and the reference therein.

Bhaskar and Lakshmikantham [2] established some coupled fixed point theorems and apply these to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ćirić [12] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [2].

2010 *Mathematics Subject Classification*. Primary: 47H10, 54H25.

Key words and phrases. Quadruple fixed point, quadruple coincidence point, w -compatible mappings, F-weakly commuting.

Karapinar [7] introduced the concept of quadruple fixed point for single valued mappings in partially ordered metric spaces and established some quadruple fixed point theorems.

For more details on quadruple fixed point theory, see [1, 5, 6, 8, 9, 14]. Samet and Vetro [15] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for $N = 4$ (quadruple case), we have the following definition:

Definition 1.1. Let X be a non-empty set and $F : X^4 \rightarrow X$ be a given mapping. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of the mapping F if

$$\begin{aligned} F(x, y, z, w) &= x, F(y, z, w, x) = y, \\ F(z, w, x, y) &= z, F(w, x, y, z) = w. \end{aligned}$$

These concepts were extended by Deshpande and Handa [4] to multi-valued mappings and obtained quadruple coincidence points and common quadruple fixed point theorems involving hybrid pair of mappings under $\varphi - \psi$ contraction. In [4], Deshpande and Handa introduced the following for multivalued mappings:

Definition 1.2. Let X be a nonempty set, $F : X^4 \rightarrow 2^X$ (a collection of all nonempty subsets of X) be a multivalued mapping and g be a self-mapping on X . An element $(x, y, z, w) \in X^4$ is called

- (1) A quadruple fixed point of F if $x \in F(x, y, z, w)$, $y \in F(y, z, w, x)$, $z \in F(z, w, x, y)$ and $w \in F(w, x, y, z)$.
- (2) A quadruple coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y, z, w)$, $g(y) \in F(y, z, w, x)$, $g(z) \in F(z, w, x, y)$ and $g(w) \in F(w, x, y, z)$.
- (3) A common quadruple fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y, z, w)$, $y = g(y) \in F(y, z, w, x)$, $z = g(z) \in F(z, w, x, y)$ and $w = g(w) \in F(w, x, y, z)$.

We denote the set of quadruple coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x, y, z, w) \in C\{F, g\}$, then (y, z, w, x) , (z, w, x, y) and (w, x, y, z) are also in $C\{F, g\}$.

Definition 1.3. Let $F : X^4 \rightarrow 2^X$ be a multivalued mapping and g be a self-mapping on X . The hybrid pair $\{F, g\}$ is called w -compatible if $g(F(x, y, z, w)) \subseteq F(gx, gy, gz, gw)$ whenever $(x, y, z, w) \in C\{F, g\}$.

Definition 1.4. Let $F : X^4 \rightarrow 2^X$ be a multivalued mapping and g be a self-mapping on X . The mapping g is called F -weakly commuting at some point $(x, y, z, w) \in X^4$ if $g^2x \in F(gx, gy, gz, gw)$, $g^2y \in F(gy, gz, gw, gx)$, $g^2z \in F(gz, gw, gx, gy)$ and $g^2w \in F(gw, gx, gy, gz)$.

Lemma 1.1. Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.

In this paper, we establish a quadruple coincidence and common quadruple fixed point theorem for hybrid pair of mappings satisfying new contractive condition. It is to be noted that to find quadruple coincidence points, we do not use the condition of continuity of any mapping involved. Our result improve, extend, and generalize the results of Bhaskar and Lakshmikantham [2] and Lakshmikantham and Ćirić [12]. An example to validate our result has also been demonstrated.

2. MAIN RESULTS

Let Φ denote the set of all functions $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying

(i $_{\varphi}$) φ is non-decreasing,

(ii $_{\varphi}$) $\varphi(t) < t$ for all $t > 0$,

(iii $_{\varphi}$) $\lim_{r \rightarrow t^+} \varphi(r) < t$ for all $t > 0$

and Ψ denote the set of all functions $\psi : [0, +\infty) \rightarrow [0, 1)$ which satisfies

(i $_{\psi}$) ψ is continuous,

(ii $_{\psi}$) $\psi(t) < t$ for all $t > 0$.

Note that, by (i $_{\psi}$) and (ii $_{\psi}$) we have that $\psi(t) = 0$ if and only if $t = 0$.

For simplicity, we define the following:

$$(A) \quad \begin{aligned} & M(x, y, z, w, p, q, r, s) \\ &= \min \left\{ \begin{array}{l} D(gx, F(x, y, z, w)), D(gp, F(p, q, r, s)), \\ \dots, D(gw, F(w, x, y, z)), D(gs, F(s, p, q, r)), \\ D(gx, F(p, q, r, s)), D(gp, F(x, y, z, w)), \\ \dots, D(gw, F(s, p, q, r)), D(gs, F(w, x, y, z)) \end{array} \right\}, \end{aligned}$$

$$(B) \quad \begin{aligned} & m(x, y, z, w, p, q, r, s) \\ &= \min \left\{ \begin{array}{l} D(x, F(x, y, z, w)), D(p, F(p, q, r, s)), \\ \dots, D(w, F(w, x, y, z)), D(s, F(s, p, q, r)), \\ D(x, F(p, q, r, s)), D(p, F(x, y, z, w)), \\ \dots, D(w, F(s, p, q, r)), D(s, F(w, x, y, z)) \end{array} \right\}. \end{aligned}$$

Theorem 2.1. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$(1) \quad \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx, gp), d(gy, gq), \\ d(gz, gr), d(gw, gs) \end{array} \right\} \right] + \psi [M(x, y, z, w, p, q, r, s)],$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have

a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.
- (d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be arbitrary. Then $F(x_0, y_0, z_0, w_0), F(y_0, z_0, w_0, x_0), F(z_0, w_0, x_0, y_0)$ and $F(w_0, x_0, y_0, z_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0, z_0, w_0), gy_1 \in F(y_0, z_0, w_0, x_0), gz_1 \in F(z_0, w_0, x_0, y_0)$ and $gw_1 \in F(w_0, x_0, y_0, z_0)$, because $F(X^4) \subseteq g(X)$. Since $F : X^4 \rightarrow CB(X)$, therefore by Lemma 1.1, there exist $u_1 \in F(x_1, y_1, z_1, w_1), u_2 \in F(y_1, z_1, w_1, x_1), u_3 \in F(z_1, w_1, x_1, y_1)$ and $u_4 \in F(w_1, x_1, y_1, z_1)$ such that

$$\begin{aligned} d(gx_1, u_1) &\leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)), \\ d(gy_1, u_2) &\leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)), \\ d(gz_1, u_3) &\leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)), \\ d(gw_1, u_4) &\leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)). \end{aligned}$$

Since $F(X^4) \subseteq g(X)$, there exist $x_2, y_2, z_2, w_2 \in X$ such that $u_1 = gx_2, u_2 = gy_2, u_3 = gz_2$ and $u_4 = gw_2$. Thus

$$\begin{aligned} d(gx_1, gx_2) &\leq H(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)), \\ d(gy_1, gy_2) &\leq H(F(y_0, z_0, w_0, x_0), F(y_1, z_1, w_1, x_1)), \\ d(gz_1, gz_2) &\leq H(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)), \\ d(gw_1, gw_2) &\leq H(F(w_0, x_0, y_0, z_0), F(w_1, x_1, y_1, z_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n, z_n, w_n), gy_{n+1} \in F(y_n, z_n, w_n, x_n), gz_{n+1} \in F(z_n, w_n, x_n, y_n)$ and $gw_{n+1} \in F(w_n, x_n, y_n, z_n)$ such

that

$$\begin{aligned}
 & d(gx_n, gx_{n+1}) \\
 & \leq H(F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), F(x_n, y_n, z_n, w_n)) \\
 & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right] \\
 & \quad + \psi [M(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}, x_n, y_n, z_n, w_n)] \\
 & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right].
 \end{aligned}$$

Thus

$$d(gx_n, gx_{n+1}) \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right].$$

Similarly

$$\begin{aligned}
 d(gy_n, gy_{n+1}) & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right], \\
 d(gz_n, gz_{n+1}) & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right], \\
 d(gw_n, gw_{n+1}) & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right].
 \end{aligned}$$

Combining them, we get

$$\begin{aligned}
 (2) \quad & \max \left\{ \begin{array}{l} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array} \right\} \\
 & \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right],
 \end{aligned}$$

which implies, by (ii_φ) , that

$$\begin{aligned}
 & \max \left\{ \begin{array}{l} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array} \right\} \\
 & < \max \left\{ \begin{array}{l} d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n) \end{array} \right\}.
 \end{aligned}$$

This shows that the sequence $\{\delta_n\}_{n=0}^\infty$ defined by $\delta_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\}$ is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$(3) \quad \lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \max \left\{ \begin{array}{l} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array} \right\} = \delta.$$

We shall prove that $\delta = 0$. Suppose that $\delta > 0$. Letting $n \rightarrow \infty$ in (2), by using (3) and (iii_φ) , we get

$$\delta \leq \lim_{n \rightarrow \infty} \varphi(\delta_n) = \lim_{\delta_n \rightarrow \delta^+} \varphi(\delta_n) < \delta,$$

which is a contradiction. Hence

$$(4) \quad \lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \max \left\{ \begin{array}{l} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array} \right\} = 0.$$

We now prove that $\{gx_n\}_{n=0}^\infty$, $\{gy_n\}_{n=0}^\infty$, $\{gz_n\}_{n=0}^\infty$ and $\{gw_n\}_{n=0}^\infty$ are Cauchy sequences in (X, d) . Suppose, to the contrary, that one of the sequences is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}_{n=0}^\infty$, $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}_{n=0}^\infty$, $\{gz_{n(k)}\}$, $\{gz_{m(k)}\}$ of $\{gz_n\}_{n=0}^\infty$ and $\{gw_{n(k)}\}$, $\{gw_{m(k)}\}$ of $\{gw_n\}_{n=0}^\infty$ such that

$$(5) \quad \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), \\ d(gz_{n(k)}, gz_{m(k)}), d(gw_{n(k)}, gw_{m(k)}) \end{array} \right\} \geq \varepsilon, \quad k = 1, 2, \dots$$

We can choose $n(k)$ to be the smallest positive integer satisfying (5), so

$$(6) \quad \max \left\{ \begin{array}{l} d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)}), \\ d(gz_{n(k)-1}, gz_{m(k)}), d(gw_{n(k)-1}, gw_{m(k)}) \end{array} \right\} < \varepsilon.$$

By (5) and (6) and the triangle inequality, we have

$$\begin{aligned} \varepsilon \leq r_k &= \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), \\ d(gz_{n(k)}, gz_{m(k)}), d(gw_{n(k)}, gw_{m(k)}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1}), \\ d(gz_{n(k)}, gz_{n(k)-1}), d(gw_{n(k)}, gw_{n(k)-1}) \end{array} \right\} \\ &\quad + \max \left\{ \begin{array}{l} d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)}), \\ d(gz_{n(k)-1}, gz_{m(k)}), d(gw_{n(k)-1}, gw_{m(k)}) \end{array} \right\} \\ &< \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1}), \\ d(gz_{n(k)}, gz_{n(k)-1}), d(gw_{n(k)}, gw_{n(k)-1}) \end{array} \right\} + \varepsilon. \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality and using (4), we get

$$(7) \quad \lim_{k \rightarrow \infty} r_k = \lim_{k \rightarrow \infty} \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), \\ d(gz_{n(k)}, gz_{m(k)}), d(gw_{n(k)}, gw_{m(k)}) \end{array} \right\} = \varepsilon.$$

By the triangle inequality, we have

$$\begin{aligned} r_k &= \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), \\ d(gz_{n(k)}, gz_{m(k)}), d(gw_{n(k)}, gw_{m(k)}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{n(k)+1}), d(gy_{n(k)}, gy_{n(k)+1}), \\ d(gz_{n(k)}, gz_{n(k)+1}), d(gw_{n(k)}, gw_{n(k)+1}) \end{array} \right\} \\ &\quad + \max \left\{ \begin{array}{l} d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), \\ d(gz_{n(k)+1}, gz_{m(k)+1}), d(gw_{n(k)+1}, gw_{m(k)+1}) \end{array} \right\} \\ &\quad + \max \left\{ \begin{array}{l} d(gx_{m(k)+1}, gx_{m(k)}), d(gy_{m(k)+1}, gy_{m(k)}), \\ d(gz_{m(k)+1}, gz_{m(k)}), d(gw_{m(k)+1}, gw_{m(k)}) \end{array} \right\} \\ &\leq \delta_{n(k)} + \delta_{m(k)} + \max \left\{ \begin{array}{l} d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), \\ d(gz_{n(k)+1}, gz_{m(k)+1}), d(gw_{n(k)+1}, gw_{m(k)+1}) \end{array} \right\}. \end{aligned}$$

Thus

$$(8) \quad r_k \leq \delta_{n(k)} + \delta_{m(k)} + \max \left\{ \begin{array}{l} d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), \\ d(gz_{n(k)+1}, gz_{m(k)+1}), d(gw_{n(k)+1}, gw_{m(k)+1}) \end{array} \right\}.$$

Since

$$\begin{aligned} gx_{n(k)+1} &\in F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), \\ gx_{m(k)+1} &\in F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}), \\ gy_{n(k)+1} &\in F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), \\ gy_{m(k)+1} &\in F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)}), \\ gz_{n(k)+1} &\in F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), \\ gz_{m(k)+1} &\in F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}), \\ gw_{n(k)+1} &\in F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}) \quad \text{and} \\ gw_{m(k)+1} &\in F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}), \end{aligned}$$

therefore by (1) and triangle inequality, we have

$$\begin{aligned} &d(gx_{n(k)+1}, gx_{m(k)+1}) \\ &\leq H(F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)})) \\ &\leq \varphi \left[\max \left\{ \begin{array}{l} d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), \\ d(gz_{n(k)}, gz_{m(k)}), d(gw_{n(k)}, gw_{m(k)}) \end{array} \right\} \right] \\ &\quad + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right] \\ &\leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right]. \end{aligned}$$

Thus

$$d(gx_{n(k)+1}, gx_{m(k)+1}) \leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right].$$

Similarly

$$\begin{aligned} d(gy_{n(k)+1}, gy_{m(k)+1}) &\leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right], \\ d(gz_{n(k)+1}, gz_{m(k)+1}) &\leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right], \\ d(gw_{n(k)+1}, gw_{m(k)+1}) &\leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right]. \end{aligned}$$

Combining them, we get

$$(9) \quad \begin{aligned} & \max \left\{ \begin{array}{l} d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), \\ d(gz_{n(k)+1}, gz_{m(k)+1}), d(gw_{n(k)+1}, gw_{m(k)+1}) \end{array} \right\} \\ & \leq \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right]. \end{aligned}$$

By (8) and (9), we get

$$r_k \leq \delta_{n(k)} + \delta_{m(k)} + \varphi(r_k) + \psi \left[M \left(\begin{array}{l} x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}, \\ x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)} \end{array} \right) \right].$$

Letting $k \rightarrow \infty$ in the above inequality, by using (4), (7), (A), (i_ψ) , (ii_ψ) and (iii_φ) , we get

$$\varepsilon \leq 0 + 0 + \lim_{k \rightarrow \infty} \varphi(r_k) + 0 \leq \lim_{r_k \rightarrow \varepsilon^+} \varphi(r_k) < \varepsilon,$$

which is a contradiction. This shows that $\{gx_n\}_{n=0}^\infty$, $\{gy_n\}_{n=0}^\infty$, $\{gz_n\}_{n=0}^\infty$ and $\{gw_n\}_{n=0}^\infty$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, therefore there exist $x, y, z, w \in X$ such that

$$(10) \quad \begin{aligned} \lim_{n \rightarrow \infty} gx_n &= gx, & \lim_{n \rightarrow \infty} gy_n &= gy, \\ \lim_{n \rightarrow \infty} gz_n &= gz & \text{and} & \lim_{n \rightarrow \infty} gw_n &= gw. \end{aligned}$$

Now, since $gx_{n+1} \in F(x_n, y_n, z_n, w_n)$, $gy_{n+1} \in F(y_n, z_n, w_n, x_n)$, $gz_{n+1} \in F(z_n, w_n, x_n, y_n)$ and $gw_{n+1} \in F(w_n, x_n, y_n, z_n)$, therefore by using condition (1), we get

$$\begin{aligned} & D(gx_{n+1}, F(x, y, z, w)) \\ & \leq H(F(x_n, y_n, z_n, w_n), F(x, y, z, w)) \\ & \leq \varphi [\max \{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), d(gw_n, gw)\}] \\ & \quad + \psi [M(x_n, y_n, z_n, w_n, x, y, z, w)]. \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality, by using (10), (A), (i_ψ) , (ii_ψ) and (iii_φ) , we get

$$D(gx, F(x, y, z, w)) \leq \lim_{t \rightarrow 0^+} \varphi(t) + 0 = 0 + 0 = 0,$$

which implies that

$$D(gx, F(x, y, z, w)) = 0.$$

Similarly, we can get

$$\begin{aligned} D(gy, F(y, z, w, x)) &= 0, \\ D(gz, F(z, w, x, y)) &= 0, \\ D(gw, F(w, x, y, z)) &= 0, \end{aligned}$$

it follows that

$$\begin{aligned} gx &\in F(x, y, z, w), \quad gy \in F(y, z, w, x), \\ gz &\in F(z, w, x, y), \quad gw \in F(w, x, y, z), \end{aligned}$$

that is, (x, y, z, w) is a quadruple coincidence point of F and g .

Suppose now that (a) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$

$$(11) \quad \lim_{n \rightarrow \infty} g^n x = p, \quad \lim_{n \rightarrow \infty} g^n y = q, \quad \lim_{n \rightarrow \infty} g^n z = r \quad \text{and} \quad \lim_{n \rightarrow \infty} g^n w = s,$$

where $p, q, r, s \in X$. Since g is continuous at p, q, r and s , we have, by (11), that p, q, r and s are fixed points of g , that is,

$$(12) \quad gp = p, \quad gq = q, \quad gr = r \quad \text{and} \quad gs = s.$$

As F and g are w -compatible, so for all $n \geq 1$

$$(13) \quad \begin{aligned} g^n x &\in F(g^{n-1}x, g^{n-1}y, g^{n-1}z, g^{n-1}w), \\ g^n y &\in F(g^{n-1}y, g^{n-1}z, g^{n-1}w, g^{n-1}x), \\ g^n z &\in F(g^{n-1}z, g^{n-1}w, g^{n-1}x, g^{n-1}y), \\ g^n w &\in F(g^{n-1}w, g^{n-1}x, g^{n-1}y, g^{n-1}z). \end{aligned}$$

By using (1) and (13), we obtain

$$\begin{aligned} &D(g^n x, F(p, q, r, s)) \\ &\leq H(F(g^{n-1}x, g^{n-1}y, g^{n-1}z, g^{n-1}w), F(p, q, r, s)) \\ &\leq \varphi [\max \{d(g^n x, gp), d(g^n y, gq), d(g^n z, gr), d(g^n w, gs)\}] \\ &\quad + \psi [M(g^{n-1}x, g^{n-1}y, g^{n-1}z, g^{n-1}w, p, q, r, s)]. \end{aligned}$$

On taking limit as $n \rightarrow \infty$ in the above inequality, by using (11), (12), (A), (i_ψ) , (ii_ψ) and (iii_φ) , we get

$$D(gp, F(p, q, r, s)) \leq \lim_{t \rightarrow 0^+} \varphi(t) + 0 = 0 + 0 = 0,$$

which implies that

$$D(gp, F(p, q, r, s)) = 0.$$

Similarly we can get

$$D(gq, F(q, r, s, p)) = D(gr, F(r, s, p, q)) = D(gs, F(s, p, q, r)) = 0,$$

which implies that

$$(14) \quad \begin{aligned} gp &\in F(p, q, r, s), \quad gq \in F(q, r, s, p), \\ gr &\in F(r, s, p, q), \quad gs \in F(s, p, q, r). \end{aligned}$$

By (12) and (14), we get

$$\begin{aligned} p &= gp \in F(p, q, r, s), \\ q &= gq \in F(q, r, s, p), \\ r &= gr \in F(r, s, p, q), \\ s &= gs \in F(s, p, q, r), \end{aligned}$$

that is, (p, q, r, s) is a common quadruple fixed point of F and g .

Suppose now that (b) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$, g is F -weakly commuting, that is,

$$\begin{aligned} g^2x &\in F(gx, gy, gz, gw), & g^2y &\in F(gy, gz, gw, gx), \\ g^2z &\in F(gz, gw, gx, gy), & g^2w &\in F(gw, gx, gy, gz) \end{aligned}$$

and

$$g^2x = gx, \quad g^2y = gy, \quad g^2z = gz, \quad g^2w = gw.$$

Thus

$$\begin{aligned} gx &= g^2x \in F(gx, gy, gz, gw), & gy &= g^2y \in F(gy, gz, gw, gx), \\ gz &= g^2z \in F(gz, gw, gx, gy), & gw &= g^2w \in F(gw, gx, gy, gz), \end{aligned}$$

that is, (gx, gy, gz, gw) is a common quadruple fixed point of F and g .

Suppose now that (c) holds. Assume that for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$, $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$. Since g is continuous at x, y, z and w , therefore x, y, z and w are fixed points of g , that is, $gx = x$, $gy = y$, $gz = z$ and $gw = w$. Since $(x, y, z, w) \in C\{F, g\}$. Therefore, we get $x = gx \in F(x, y, z, w)$, $y = gy \in F(y, z, w, x)$, $z = gz \in F(z, w, x, y)$, $w = gw \in F(w, x, y, z)$, that is, (x, y, z, w) is a common quadruple fixed point of F and g .

Finally, suppose that (d) holds. Let $g(C\{F, g\}) = \{(x, x, x, x)\}$. Then $\{x\} = \{gx\} = F(x, x, x, x)$. Hence (x, x, x, x) is quadruple fixed point of F and g . \square

Example 2.1. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \rightarrow [0, +\infty)$ defined by $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \times X \rightarrow CB(X)$ be defined as

$$F(x, y, z, w) = \begin{cases} \{0\}, & \text{for } x, y, z, w = 1 \\ \left[0, \frac{x^2 + y^2 + z^2 + w^2}{8}\right], & \text{for } x, y, z, w \in [0, 1) \end{cases}$$

and $g : X \rightarrow X$ be defined as

$$g(x) = x^2, \quad \text{for all } x \in X.$$

Define $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2}, & \text{for } t \neq 1 \\ \frac{3}{4}, & \text{for } t = 1 \end{cases}$$

and $\psi : [0, +\infty) \rightarrow [0, 1)$ by

$$\psi(t) = \frac{t}{4}, \text{ for } t \geq 0.$$

Now, for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w, p, q, r, s \in [0, 1)$, we have **Case (a)**. If $x^2 + y^2 + z^2 + w^2 = p^2 + q^2 + r^2 + s^2$, then

$$\begin{aligned} & H(F(x, y, z, w), F(p, q, r, s)) \\ &= \frac{p^2 + q^2 + r^2 + s^2}{8} \\ &\leq \frac{1}{8} \max\{x^2, p^2\} + \frac{1}{8} \max\{y^2, q^2\} + \frac{1}{8} \max\{z^2, r^2\} + \frac{1}{8} \max\{w^2, s^2\} \\ &\leq \frac{1}{8} d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs) \\ &\leq \frac{1}{8} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\ &\quad + \frac{1}{8} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\ &\quad + \frac{1}{8} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\ &\quad + \frac{1}{8} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\ &\leq \frac{1}{2} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\ &\leq \varphi [\max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}] \\ &\quad + \psi [M(x, y, z, w, p, q, r, s)]. \end{aligned}$$

Case (b). If $x^2 + y^2 + z^2 + w^2 \neq p^2 + q^2 + r^2 + s^2$ with $x^2 + y^2 + z^2 + w^2 < p^2 + q^2 + r^2 + s^2$, then

$$\begin{aligned} & H(F(x, y, z, w), F(p, q, r, s)) \\ &= \frac{p^2 + q^2 + r^2 + s^2}{8} \\ &\leq \frac{1}{8} \max\{x^2, p^2\} + \frac{1}{8} \max\{y^2, q^2\} + \frac{1}{8} \max\{z^2, r^2\} + \frac{1}{8} \max\{w^2, s^2\} \\ &\leq \frac{1}{8} d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs) \\ &\leq \frac{1}{8} \max\{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& \leq \frac{1}{2} [\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}] \\
& \leq \varphi [\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}] \\
& \quad + \psi [M(x, y, z, w, p, q, r, s)].
\end{aligned}$$

Similarly, we obtain the same result for $p^2 + q^2 + r^2 + s^2 < x^2 + y^2 + z^2 + w^2$. Thus the contractive condition (1) is satisfied for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w, p, q, r, s \in [0, 1)$. Again, for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w \in [0, 1)$ and $p, q, r, s = 1$, we have

$$\begin{aligned}
& H(F(x, y, z, w), F(p, q, r, s)) \\
& = \frac{x^2 + y^2 + z^2 + w^2}{8} \\
& \leq \frac{1}{8} \max\{x^2, p^2\} + \frac{1}{8} \max\{y^2, q^2\} + \frac{1}{8} \max\{z^2, r^2\} + \frac{1}{8} \max\{w^2, s^2\} \\
& \leq \frac{1}{8} d(gx, gp) + \frac{1}{8} d(gy, gq) + \frac{1}{8} d(gz, gr) + \frac{1}{8} d(gw, gs) \\
& \leq \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& \quad + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& \quad + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& \quad + \frac{1}{8} \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\} \\
& \leq \frac{1}{2} [\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}] \\
& \leq \varphi [\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}] \\
& \quad + \psi [M(x, y, z, w, p, q, r, s)].
\end{aligned}$$

Thus the contractive condition (1) is satisfied for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w \in [0, 1)$ and $p, q, r, s = 1$. Similarly, we can see that the contractive condition (1) is satisfied for all $x, y, z, w, p, q, r, s \in X$ with $x, y, z, w, p, q, r, s = 1$. Hence, the hybrid pair $\{F, g\}$ satisfies the contractive condition (1), for all $x, y, z, w, p, q, r, s \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0, 0, 0, 0)$ is a common quadruple

fixed point of hybrid pair $\{F, g\}$. The function $F : X^4 \rightarrow CB(X)$ involved in this example is not continuous on X^4 .

Corollary 2.1. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$\begin{aligned} & H(F(x, y, z, w), F(p, q, r, s)) \\ & \leq \varphi \left[\frac{d(gx, gp) + d(gy, gq) + d(gz, gr) + d(gw, gs)}{4} \right] \\ & \quad + \psi [M(x, y, z, w, p, q, r, s)], \end{aligned}$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.
- (d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

Proof. It suffices to remark that

$$\begin{aligned} & \frac{d(gx, gp) + d(gy, gq) + d(gz, gr) + d(gw, gs)}{4} \\ & \leq \frac{\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}}{4} \\ & \quad + \frac{\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}}{4} \\ & \quad + \frac{\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}}{4} \\ & \quad + \frac{\max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}}{4} \\ & \leq \max \{d(gx, gp), d(gy, gq), d(gz, gr), d(gw, gs)\}. \end{aligned}$$

Then, we apply Theorem 2.1, since φ is non-decreasing. \square

If we put $g = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.2. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$\begin{aligned} & H(F(x, y, z, w), F(p, q, r, s)) \\ & \leq \varphi [\max \{d(x, p), d(y, q), d(z, r), d(w, s)\}] \\ & \quad + \psi [m(x, y, z, w, p, q, r, s)], \end{aligned}$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Then F has a quadruple fixed point.

If we put $g = I$ (the identity mapping) in the Corollary 2.1, we get the following result:

Corollary 2.3. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$\begin{aligned} & H(F(x, y, z, w), F(p, q, r, s)) \\ & \leq \varphi \left[\frac{d(x, p) + d(y, q) + d(z, r) + d(w, s)}{4} \right] \\ & \quad + \psi [m(x, y, z, w, p, q, r, s)], \end{aligned}$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Then F has a quadruple fixed point.

If we put $\psi(t) = 0$ in Theorem 2.1, we get the following result:

Corollary 2.4. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq \varphi \left[\max \left\{ \begin{array}{l} d(gx, gp), d(gy, gq), \\ d(gz, gr), d(gw, gs) \end{array} \right\} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.
- (d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $\psi(t) = 0$ in Corollary 2.1, we get the following result:

Corollary 2.5. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq \varphi \left[\frac{d(gx, gp) + d(gy, gq) + d(gz, gr) + d(gw, gs)}{4} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.
- (d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.4, we get the following result:

Corollary 2.6. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq \varphi \left[\max \left\{ \begin{array}{l} d(x, p), d(y, q), \\ d(z, r), d(w, s) \end{array} \right\} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$. Then F has a quadruple fixed point.

If we put $g = I$ (the identity mapping) in the Corollary 2.5, we get the following result:

Corollary 2.7. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq \varphi \left[\frac{d(x, p) + d(y, q) + d(z, r) + d(w, s)}{4} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $\varphi \in \Phi$. Then F has a quadruple fixed point.

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Corollary 2.4, we get the following result:

Corollary 2.8. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq k \max \left\{ \begin{array}{l} d(gx, gp), d(gy, gq), \\ d(gz, gr), d(gw, gs) \end{array} \right\},$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.
- (d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Corollary 2.5, we get the following result:

Corollary 2.9. *Let (X, d) be a metric space. Assume $F : X^4 \rightarrow CB(X)$ and $g : X \rightarrow X$ be two mappings satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq k \left[\frac{d(gx, gp) + d(gy, gq) + d(gz, gr) + d(gw, gs)}{4} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Furthermore assume that $F(X^4) \subseteq g(X)$ and $g(X)$ is a complete subset of X . Then F and g have a quadruple coincidence point. Moreover, F and g have a common quadruple fixed point, if one of the following conditions holds:

- (a) F and g are w -compatible. $\lim_{n \rightarrow \infty} g^n x = p$, $\lim_{n \rightarrow \infty} g^n y = q$, $\lim_{n \rightarrow \infty} g^n z = r$ and $\lim_{n \rightarrow \infty} g^n w = s$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$ and g is continuous at p, q, r and s .
- (b) g is F -weakly commuting for some $(x, y, z, w) \in C\{F, g\}$ and gx, gy, gz and gw are fixed points of g , that is, $g^2x = gx, g^2y = gy, g^2z = gz$ and $g^2w = gw$.
- (c) g is continuous at x, y, z and w . $\lim_{n \rightarrow \infty} g^n p = x$, $\lim_{n \rightarrow \infty} g^n q = y$, $\lim_{n \rightarrow \infty} g^n r = z$ and $\lim_{n \rightarrow \infty} g^n s = w$ for some $(x, y, z, w) \in C\{F, g\}$ and for some $p, q, r, s \in X$.

(d) $g(C\{F, g\})$ is a singleton subset of $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.8, we get the following result:

Corollary 2.10. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq k \max \{d(x, p), d(y, q), d(z, r), d(w, s)\},$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Then F has a quadruple fixed point.

If we put $g = I$ (the identity mapping) in the Corollary 2.9, we get the following result:

Corollary 2.11. *Let (X, d) be a complete metric space, $F : X^4 \rightarrow CB(X)$ be a mapping satisfying*

$$H(F(x, y, z, w), F(p, q, r, s)) \leq k \left[\frac{d(x, p) + d(y, q) + d(z, r) + d(w, s)}{4} \right],$$

for all $x, y, z, w, p, q, r, s \in X$, where $0 < k < 1$. Then F has a quadruple fixed point.

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