Mathematica Moravica
Vol. **16-2** (2012), 99–106

A Slant Helix Characterization in Riemann-Otsuki Space

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Abstract. In this paper, we establish new characterization for a slant helix from the view point of Riemann-Otsuki space.

Introduction

One of the principal goals of the study of curve theories is to develop new and alternative characterization for helices. Simply a helix in Euclidean 3-space is a curve of constant slope. The study of this curve dates from 1802 with Lancret's well-known statement"A curve is a helix if and only if ratio of curvature to torsion is constant" [6]. From past to nowadays numerous scientists have studied on this magic subject from different point of view, [1, 7, 2, 5, 9].

On the other hand Otsuki spaces, firstly introduced by T. Otsuki and A. Moor, have interesting properties because of supposing the relation

$$\nabla_k g_{ij} = \gamma_k g_{ij}$$

where g_{ij} and γ_k denote Riemannian metric tensor and a recurrance tensor, respectively. If $\nabla_k g_{ij} = 0$ holds, the space called Riemann-Otsuki space. Riemann-Otsuki space is studied by various authors by different aspects [3, 4, 8].

In [3] the author considered Riemann-Otsuki space and determined the Frenet formula with respect to the covariant and contravariant part of the connection.

The present study deals with the helices of special type in Riemann-Otsuki space. Making use of this Frenet formula we obtain new characterizations for a slant helix in Riemann-Otsuki space.

2010 *Mathematics Subject Classification.* 53B05, 53B15, 53B30.

Key words and phrases. Riemann-Otsuki space, Frenet Formula, slant helix.

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1. Preliminaries

In Otsuki spaces,we are given a-priori given tensor P_j^i such that det $\left\|P_j^i\right\| \neq 0$ holds. Q_j^i is the inverse tensor that satisfies $P_j^i Q_r^j = \delta_r^i$ with respect to the local coordinates x^i of an n-dimensional differentiable manifold,

In addition for metric Otsuki spaces the metric tensor g_{ij} (det $\|g_{ij}\| \neq 0$) is given by a relation for $\nabla_k g_{ij}$. In W-O$_n$ (Weyl-Otsuki space) this relation satisfies

$$\nabla_k g_{ij} = \gamma_k g_{ij},$$

but in $R - O_n$ (Riemann-Otsuki space)

$$\nabla_k g_{ij} = 0 \tag{1.1}$$

holds. In Otsuki spaces the covariant differential of the tensor T_j^i is defined by

$$DT_j^i = P_a^i P_b^j \overline{D} T_b^a = P_a^i P_b^j \left(\partial_k T_b^a +' \Gamma_{rk}^a T_b^r -'' \Gamma_{bk}^r T_r^a \right) dx^k. \tag{1.2}$$

The Leibnitz formula does not hold for this differential. Here we denote the basic covariant differential by $\overline{D}$. The characteristic of the Otsuki spaces are the different coefficients.We may show it as follows

$$\delta_j^i \,|_k =' \Gamma_{jk}^i -'' \Gamma_{jk}^i \neq 0. \tag{1.3}$$

The coefficient of the connection $''\Gamma_{jk}^i$ is determined from the relation (1.1). For the coefficients of connection $'\Gamma_{jk}^i$ we use the following equation which is known as Otsuki's relation;

$$\partial_k P_j^i +'' \Gamma_{ak}^i P_j^a - P_a^{i\prime} \Gamma_{jk}^a = 0. \tag{1.4}$$

In Otsuki spaces it is possible to determine the covariant differentials D and $\overline{D}$ with respect only covariant and contravariant parts of the connection. So

$$'\overline{D} T_j^i = \nabla_k T_j^i dx^k = \left(\partial_k T_j^i +' \Gamma_{rk}^i T_j^r -' \Gamma_{jk}^r T_r^i \right) dx^k \tag{1.5}$$

holds. For this basic covariant differential the Leibnitz formula holds.The basic covariant differential $''\overline{D}$ can be defined in the same way.

It is characteristic that the basic covariant differential $'\overline{D}$ is identical in the case of contravariant indices with the basic covariant differential $\overline{D}$, and similarly in the case of covariant indices the basic covariant differential $''\overline{D}$ is identical with the basic covariant differential $\overline{D}$.

We give the basic relations as follows [3]

$$'\overline{D}g_{ij} = dg_{ij} - \left({'\Gamma^r_{ik}} g_{rj} +' \Gamma^r_{jk} g_{ir}\right) dx^k, \tag{1.6}$$

$$''\overline{D}g_{ij} = dg_{ij} - \left({''\Gamma^r_{ik}} g_{rj} +'' \Gamma^r_{jk} g_{ir}\right) dx^k = 0, \tag{1.7}$$

$$'\overline{D}g^{ra} = -g^{ia} g^{jr} \left('\overline{D}g_{ij}\right), \tag{1.8}$$

$$''\overline{D}g^{ra} = 0. \tag{1.9}$$

In [3], the author considered the Frenet formula with respect to the contravariant and covariant components of the vectors in detail. In addition the Frenet formula for the covariant part is not different from the well-known formula of the Riemannian case. Hence we only consider the contravariant components of the vectors for Riemann-Otsuki space. For a deeper understanding we refer to [3].

1.1. **The Frenet Formula with Respect to the Contravariant Components of the Vectors.** We can take a point M of the curve $C : x^i = x^i(s)$ where s is the arclength parameter. For point M the components of the unit tangent vector v_0 are given as $v_0^i = \frac{dx^i}{ds}$.

Theorem 1.1. *If $C : x^i(s)$ is the curve of an $R - O_n$ space and v_l, $l = 0, ..., p-1$ $(p < n)$ are mutually orthogonal unit vectors which satisfy the following relation*

$$\overline{D}v_l j = -\kappa_l v^j_{l-1} + \kappa_{l+1} v^j_{l+1} + v^r_l \overline{D}\delta^j_r \tag{1.10}$$

such that

$$\begin{cases} \kappa_q = 0, & \textit{for } q = 0, \\ \kappa_q = \left(g_{ij}(\overline{D}_{q-1} v^j + \kappa_{q-1} v^j_{q-2})\right), & \textit{for } q = 1, \ldots, p-1. \end{cases} \tag{1.11}$$

Here v_{p+1} is the unit vector orthogonal to all before and $\kappa_0 = 0$ and $\kappa_n = 0$ holds. Then the vector v_p satisfies the relation, (1.10), too.

If we use Otsuki's covariant differential D, then from the connection $D^j_v = P^j_a \overline{D}v^a$ it follows that $\overline{D}v^a = Q^a_i Dv^i$. Applying this on (1.10), we get

$$Dv_l\ j = P^j_i(-\kappa_l v^i_{l-1} + \kappa_{l+1} v^i_{l+1}) + v^q_l Q^b_q D\delta^j_b \tag{1.12}$$

with respect to $l = 0, \ldots, n-1$; $\kappa_0 = 0$; $\kappa_n = 0$. Then we can now state the following theorem [3].

Theorem 1.2. *Let us take a point M of the curve C in the $R - O_n$ space and $v_0, v_1, \ldots, v_{n-1}$ are the mutually orthogonal unit vectors that satisfy the relations* (1.10) *and* (1.11) *so that $\kappa_0 = 0$ and $\kappa_n = 0$ hold. Hence we obtain* (1.12) *by applying the covariant differential D on the covariant components of the observed vectors* [3].

Remark 1.1.1. The relation (1.12) is the Frenet formula with respect to the covariant differential $'D$, applied on the contravariant components of the vectors.

If we apply the basic covariant differential $''\overline{D}$ to the tangent vectors v_0^i and v_0^j, we obtain the known Frenet formulas for the Riemannian case so we omit this version of the study.

2. Frenet Formula in $R - O_3$ and Slant Helices

According to observation above, we may express the following theorem for dimension 3 by a close analogy with Theorem 1.1

Theorem 2.1. *If $C : x^3(s)$ is the curve of an R-O_3 space and v_l, $l = 0, 1, 2$ are mutually orthogonal unit vectors which satisfies*

$$Dv_l j = P_i^j(-\kappa_l v_{l-1}^i + \kappa_{l+1} v_{l+1}^i) + v_l^q Q_q^b D\delta_b^j \tag{2.1}$$

and v_4 is the unit orthogonal to all before and $\kappa_0 = 0$, $\kappa_3 = 0$ holds then the vector v_3 satisfies equation (2.1)*, too.*

Thus we obtain Frenet trihedron as follows

$$\begin{aligned} Dv_0\ j &= P_i^j(\kappa_1 v_1^i) + v_0^q Q_q^b D\delta_b^j, \\ Dv_1\ j &= P_i^j(-\kappa_1 v_0^i + \kappa_2 v_2^i) + v_1^q Q_q^b D\delta_b^j, \\ Dv_2\ j &= P_i^j(-\kappa_2 v_1^i) + v_2^q Q_q^b D\delta_b^j. \end{aligned} \tag{2.2}$$

Using matrix expression we get

$$\begin{bmatrix} Dv_0 j \\ Dv_1 j \\ Dv_2 j \end{bmatrix} = P_i^j \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_0^i \\ v_1^i \\ v_2^i \end{bmatrix} + Q_q^b D\delta_b^j \begin{bmatrix} v_0^q \\ v_1^q \\ v_2^q \end{bmatrix}. \tag{2.3}$$

With these preparatory remarks we give the following

Definition 2.1. A unit speed curve α is called a slant helix if there exists a constant vector field U in R-O_3 such that the function $g_{ij}(v_1 j, U)$ is constant.

This definition is motivated by what happens in Euclidean space. Recently Izumiya and Takeuchi [9] have introduced the slant helix in Euclidean space by saying that normal lines make constant angle with a fixed direction and they characterize a slant helix if and only if the function

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \tag{2.4}$$

is constant.

In this work, we focus on this subject and obtain new characterization for 3-dimensional Riemann-Otsuki space $R - O_3$.

Firstly, we assume that α is a slant helix. Let U be the vector field such that the function $g_{ij}(v_1\ j, U) = c$ is constant. There exists smooth functions a_1, a_2 and a_3 such that

$$U = a_1(s)v_0\ j + a_2(s)v_1\ j + a_3(s)v_2 j \tag{2.5}$$

As U is constant by differentiation (2.5) together (2.3) gives

$$a_1' - P_i^j\left(\kappa_1 a_2 + a_1 Q_q^b D\delta_b^j\right) = 0, \tag{2.6}$$

$$-P_i^j\left(-\kappa_1 a_1 + \kappa_2 a_3 + a_2 Q_q^b D\delta_b^j\right) = 0, \tag{2.7}$$

$$a_3' - P_i^j\left(\kappa_2 a_2 + a_3 Q_q^b D\delta_b^j\right) = 0. \tag{2.8}$$

Using (2.6) and (2.8) we get

$$a_3 = \frac{\kappa_1}{\kappa_2}a_1 - \frac{a_2}{\kappa_2}Q_q^b D\delta_b^j. \tag{2.9}$$

Moreover

$$g_{ij}(U,U) = a_1^2 + a_2^2 + a_3^2 = \text{constant}. \tag{2.10}$$

Taking into account of (2.8), (2.9) and (2.10) together and after routine calculations we set

$$a_1^2\left(1 + \left(\frac{\kappa_1}{\kappa_2}\right)^2 - 2\frac{\kappa_1}{\kappa_2^2}Y + \frac{Y_2}{\kappa_2^2}\right) = \varepsilon m^2, \quad m > 0, \ \ \varepsilon \in \{-1, 0, 1\}$$

where Y and Y_2 are constants.

Suppose that $\varepsilon = 0$ this means that U is a constant vector and result is obvious. If $\varepsilon = \pm 1$ we get

$$a_1^2 = \pm\frac{m^2}{\left(1 + \left(\frac{\kappa_1}{\kappa_2}\right)^2 - 2\frac{\kappa_1}{\kappa_2^2}Y + \frac{Y_2}{\kappa_2^2}\right)}$$

and

$$\begin{aligned} a_1 &= \pm\frac{m}{\sqrt{\left(1 + \left(\frac{\kappa_1}{\kappa_2}\right)^2 - 2\frac{\kappa_1}{\kappa_2^2}Y + \frac{Y_2}{\kappa_2^2}\right)}}, \quad \text{or} \\ a_1 &= \pm\frac{m}{\sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)^2 - 2\frac{\kappa_1}{\kappa_2^2}Y + \frac{Y_2}{\kappa_2^2} - 1}}. \end{aligned} \tag{2.11}$$

Then (2.6) yields

$$\frac{d}{ds}\left[\pm\frac{m}{\sqrt{\left(1 + \left(\frac{\kappa_1}{\kappa_2}\right)^2 - 2\frac{\kappa_1}{\kappa_2^2}Y + \frac{Y_2}{\kappa_2^2}\right)}}\right] = P_i^j\left(\kappa_1 a_2 + a_1 Q_q^b D\delta_b^j\right) \tag{2.12}$$

or

$$\frac{d}{ds}\left[\pm\frac{m}{\sqrt{\left(\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1\right)}}\right]=-P_i^j\left(\kappa_1 a_2+a_1 Q_q^b D\delta_b^j\right) \tag{2.13}$$

on I. This can be written as

$$\frac{\left(\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1\right)'}{\left(\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1\right)^{3/2}}=\pm\frac{P_i^j\left(\kappa_1 a_2+a_1 Q_q^b D\delta_b^j\right)}{m} \tag{2.14}$$

and

$$\frac{\left(1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}\right)'}{\left(1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}\right)^{3/2}}=\pm\frac{P_i^j\left(\kappa_1 a_2+a_1 Q_q^b D\delta_b^j\right)}{m}. \tag{2.15}$$

Conversely, assume that the conditions (2.14) and (2.15) are satisfied. To simplify the calculations, we assume that (2.14) is constant, namely a_1 and κ_1. We define

$$\begin{aligned} U= &\frac{\kappa_2}{\sqrt{1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}}}v_0 j+P_i^j\left(\kappa_1 a_2+a_1 Q_q^b D\delta_b^j\right)v_1 j \\ &+\left[\frac{\kappa_1}{\sqrt{1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}}}-a_2\frac{Q_q^b D\delta_b^j}{\kappa_2}\right]v_2 j \end{aligned} \tag{2.16}$$

A differentiation of (2.16) together Frenet equations gives $\frac{dU}{ds}=0$, that is, U is a constant vector. On the other hand taking into account a_1 and κ_1 are constant $g(v_1\ j, U)=cons\tan t=c$ and this means that α is a slant helix. Hence we have proved the following theorem.

Theorem 2.2. *Let α be unit speed curve in R-O_3. Then α is a slant helix if and only if either one the next two functions*

$$\frac{\left(\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1\right)'}{\left(\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1\right)^{3/2}},$$

or

$$\frac{\left(1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}\right)'}{\left(1+\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}\right)^{3/2}}$$

is constant everywhere $\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1$ *does not vanish.*

Remark 2.1. The authors don't know what happens if

$$\left(\frac{\kappa_1}{\kappa_2}\right)^2-2\frac{\kappa_1}{\kappa_2^2}Y+\frac{Y_2}{\kappa_2^2}-1$$

vanishes in some points. On the other hand if we take covariant part of the connection the Frenet formula of the $R-O_3$ space are not different from the known Frenet formulas of 3-dimensional Euclidean space. Hence we get the same results with [9] for this type.

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