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Diametral φ-Contraction on Topological Spaces

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Abstract. This paper introduces generalization of some already known results obtained by the author during 80's. This paper extends some known results on topological spaces and describe a class of conditions sufficient for the existence of fixed points.

1. Introduction and history

Let (X, ρ) be a metric space and T a mapping of X into itself. A metric space X is said to be T-*orbitally complete* iff every Cauchy sequence which is contained in orbit $\mathcal{O}(x) = \{x, Tx, T^2, \ldots\}$ for some $x \in X$ converges in X.

In 1980 I have been proved the following result of fixed point on metric spaces which has a best long of all known sufficiently conditions for the existing of unique fixed point, cf. Tasković [3], [4] and [5]. This result introduces generalization of a great number of known results.

In [3] I have been introduced the concept of a *diametral φ-contraction T* of a metric space (X, ρ) into itself, i.e., there exists a function $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0 := [0, +\infty)$ satisfying

$$(\mathrm{I}\varphi) \qquad (\forall t \in \mathbb{R}_+ := (0, +\infty))\Big(\varphi(t) < t \quad \text{and} \quad \limsup_{z \to t+0} \varphi(z) < t\Big)$$

such that

$$\rho[Tx, Ty] \leq \varphi\Big(\operatorname{diam}\{x, y, Tx, Ty, T^2x, T^2y, \ldots\}\Big)$$

for all $x, y \in X$, where diam denoted diameter.

In [3] Tasković has proved the following result: *Let T be a diametral φ-contraction on a T-orbitally complete metric space (X, ρ). If* $\operatorname{diam} \mathcal{O}(x) \in \mathbb{R}_+^0$*, then T has a unique fixed point $\xi \in X$.*

A brief first proof of this statement may be found in Tasković [3]. Also some brief proofs for this we can see in Tasković [4], [5] and [7]. For some results in connection with this see Ohta-Nikaido [1].

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2. MAIN RESULTS

In this paper we extend the preceding result on topological spaces and we describe a class of conditions sufficient for the existence of fixed points.

Let $X := (X, A)$ be a topological space, where $T : X \to X$ and $A : X \times X \to \mathbb{R}^0_+$ is a given functional. For $S \subset X$ we denoted toptdiam(S) as a *topological diameter* of S, where

$$\operatorname{toptdiam}(S) := \sup\Big\{A(x,y) : x, y \in S\Big\};$$

and, in connection with this, a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be *topological Cauchy sequence* iff

$$\lim_{n\to\infty}\Big(\operatorname{toptdiam}\{T^n(x) : k \geq n\}\Big) = 0;$$

and, a topological space X is said to be *orbitally complete* (or T-*orbitally complete*) iff every topological Cauchy sequence which is contained in $\mathcal{O}(x)$ for some $x \in X$ converges in X. A mapping $T : X \to X$ is said to be *orbitally continuous* iff $\xi, x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x))$.

A functional f mapping X into the reals is T-*orbitally lower semicontinuous* at $p \in X$ if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $f(p) \leq \liminf_{n\to\infty} f(x_n)$.

We are now in a position to formulate the following statement, which is a roofing for a great number of known results in the fixed point theory.

Theorem 1. *Let T be a mapping of a topological space $X := (X, A)$ into itself and let X be orbitally complete. Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying (Iφ) such that*

$$\text{(D)} \qquad A(Tx, Ty) \leq \varphi\Big(\operatorname{toptdiam}\{x, y, Tx, Ty, T^2x, T^2y, \ldots\}\Big)$$

and toptdiam $\mathcal{O}(x) \in \mathbb{R}^0_+$ *for all $x, y \in X$. If $x \mapsto$* toptdiam $\mathcal{O}(x)$ *or $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous or T orbitally continuous, and $A(a, b) = 0$ implies $a = b$, then T has a unique fixed point $\xi \in X$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to ξ for every $x \in X$.*

We begin the proof with a well known lemma which is fundamental in the following context.

Lemma 1 (Tasković [8]). *Let the mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ have the property (Iφ). If the sequence (x_n) of nonnegative real numbers satisfies the condition*

$$x_{n+1} \leq \varphi(x_n), \quad n \in \mathbb{N},$$

then the sequence (x_n) tends to zero. The velocity of this convergence is not necessarily geometrical.

A brief first proof of this statement may be found in Tasković [8]. Other brief proofs for this we can see in Tasković [3], [4] and [5]. Also see Seneta [2].

Proof of Theorem 1. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ is a topological Cauchy sequence. It is easy to verify that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ satisfies the following inequality

$$\operatorname{toptdiam}\mathcal{O}(T^{n+1}x) \leq \varphi(\operatorname{toptdiam}\mathcal{O}(T^n x))$$

for $n \in \mathbb{N}$, and hence applying Lemma 2 to the sequence $(\operatorname{toptdiam}\mathcal{O}(T^n x))$ we obtain $\lim_{n\to\infty}\operatorname{toptdiam}\mathcal{O}(T^n x) = 0$. This implies that $\{T^n x\}_{n\in\mathbb{N}}$ is a topological Cauchy sequence in X and, by T-orbital completeness, there is a $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \to \operatorname{toptdiam}\mathcal{O}(x)$ is T-orbitally lower semicontinuous at ξ,

$$A(\xi, T\xi) \leq \operatorname{toptdiam}\mathcal{O}(\xi) \leq \liminf(\operatorname{toptdiam}\mathcal{O}(T^n x)) = 0;$$

thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

On the other hand, if $x \mapsto A(x, Tx)$ is a T-orbitally lower semicontinuous at ξ we have

$$A(\xi, T\xi) \leq \liminf A(T^n x, T^{n+1}x) \leq \liminf(\operatorname{toptdiam}\mathcal{O}(T^n x)) = 0;$$

and thus again $T\xi = \xi$, i.e., we have again shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T. Also, if T is orbitally continuous the proof of previous fact is trivial.

We complete the proof by showing that T can have at most one fixed point: for, if $\xi \neq \eta$ were two fixed points, then

$$\begin{aligned}
0 &< \max\{A(\xi,\eta), A(\eta,\xi)\} = \max\{A(T\xi,T\eta), A(T\eta,T\xi)\} \leq \\
&\leq \varphi\Big(\operatorname{toptdiam}\{\xi,\eta,T\xi,T\eta,T^2\xi,T^2\eta,\ldots\}\Big) = \\
&= \varphi\Big(\max\{A(\xi,\xi), A(\eta,\eta), A(\xi,\eta), A(\eta,\xi)\}\Big) = \\
&= \varphi\Big(\max\{A(\xi,\eta), A(\eta,\xi)\}\Big) < \max\{A(\xi,\eta), A(\eta,\xi)\},
\end{aligned}$$

a contradiction. The proof is complete. □

As immediate consequence of the preceding Theorem 1, we obtain directly the following interesting cases of (D):

(1) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z\to t+0}\psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leq \psi(\operatorname{toptdiam}\{x, y, Tx, Ty\})$$

for all $x, y \in X$.

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha \in [0, 1)$ such that for all $x, y \in X$ the following inequalities hold

$$A(Tx, Ty) \leq \alpha \operatorname{toptdiam}\{x, y, Tx, Ty\},$$

i.e., equivalently to

$$A(Tx, Ty) \leq \alpha \max\Big\{A(x, y), A(x, Tx), A(y, Ty), A(x, Ty), A(y, Tx)\Big\}.$$

(3) (*The condition of* $(m+k)$*-polygon*). There exists a constant $\alpha \in [0, 1)$ such that for all $x, y \in X$ the following inequality holds in the form as

$$A(Tx, Ty) \leq \alpha \operatorname{toptdiam}\Big\{x, y, Tx, Ty, \ldots, T^m x, T^k y\Big\}$$

for arbitrary fixed integers $m, k \geq 0$. (This is a linear condition for diameter of finite number of points).

(4) There exists a nondecreasing function $\psi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ satisfying the following condition in the form $\limsup_{z\to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leq \psi\Big(\operatorname{toptdiam}\{x, y, Tx, Ty, \ldots, T^m x, T^k y\}\Big)$$

for arbitrary fixed integers $m, k \geq 0$ and for all $x, y \in X$. (*This is a nonlinear condition for top.diameter of finite number of points*).

(5) There exists an increasing mapping for any coordinates of $f : (\mathbb{R}_+^0)^5 \to \mathbb{R}_+^0$ satisfying $\limsup_{z\to t+0} f(z, z, z, z, z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leq f\Big(A(x, y), A(x, Tx), A(y, Ty), A(x, Ty), A(y, Tx)\Big)$$

for all $x, y \in X$.

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 1 in the following form.

Theorem 2. *Let* T *be a mapping of a topological space* $X := (X, A)$ *into itself and let* X *be orbitally complete. Suppose that there exists a mapping* $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ *satisfying* (Iφ) *such that*

$$\operatorname{toptdiam}\{Tx, T^2x, \ldots\} \leq \varphi\Big(\operatorname{toptdiam}\{x, Tx, T^2x, \ldots\}\Big)$$

and $\operatorname{toptdiam} \mathcal{O}(x) \in \mathbb{R}_+^0$ *for every* $x \in X$. *If* $x \mapsto \operatorname{toptdiam} \mathcal{O}(x)$ *or* $x \mapsto A(x, Tx)$ *is* T*-orbitally lower semicontinuous or* T *is orbitally continuous, and* $A(a, b) = 0$ *iff* $a = b$, *then* T *has at least one fixed point in* X.

The proof of this localization statement is totally analogous with the preceding proof of Theorem 1. Thus the proof of this result we omit.

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