Mathematica Moravica
Vol. **15-1** (2011), 59–63

Diametral φ-Contraction on Topological Spaces

Milan R. Tasković

Abstract. This paper introduces generalization of some already known results obtained by the author during 80's. This paper extends some known results on topological spaces and describe a class of conditions sufficient for the existence of fixed points.

1. Introduction and history

Let (X, ρ) be a metric space and T a mapping of X into itself. A metric space X is said to be T-*orbitally complete* iff every Cauchy sequence which is contained in orbit $\mathcal{O}(x) = \{x, Tx, T^2, \dots\}$ for some $x \in X$ converges in X.

In 1980 I have been proved the following result of fixed point on metric spaces which has a best long of all known sufficiently conditions for the existing of unique fixed point, cf. Tasković [3], [4] and [5]. This result introduces generalization of a great number of known results.

In [3] I have been introduced the concept of a *diametral φ-contraction T* of a metric space (X, ρ) into itself, i.e., there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ satisfying

$$(\mathrm{I}\varphi) \qquad (\forall t \in \mathbb{R}_+ := (0, +\infty))\Big(\varphi(t) < t \quad \text{and} \quad \limsup_{z \to t+0} \varphi(z) < t\Big)$$

such that

$$\rho[Tx, Ty] \le \varphi\Big(\operatorname{diam}\{x, y, Tx, Ty, T^2x, T^2y, \dots\}\Big)$$

for all $x, y \in X$, where diam denoted diameter.

In [3] Tasković has proved the following result: *Let T be a diametral φ-contraction on a T-orbitally complete metric space (X, ρ). If* $\operatorname{diam} \mathcal{O}(x) \in \mathbb{R}^0_+$*, then T has a unique fixed point $\xi \in X$.*

A brief first proof of this statement may be found in Tasković [3]. Also some brief proofs for this we can see in Tasković [4], [5] and [7]. For some results in connection with this see Ohta-Nikaido [1].

2000 *Mathematics Subject Classification.* Primary: 47H10, 05A15; Secondary: 54H25.
Key words and phrases. Fixed point theorems, diametral φ-contraction, topological spaces, topological Cauchy sequence, topological orbital completeness, topological diameter.

2. Main results

In this paper we extend the preceding result on topological spaces and we describe a class of conditions sufficient for the existence of fixed points.

Let $X := (X, A)$ be a topological space, where $T : X \to X$ and $A : X \times X \to \mathbb{R}_+^0$ is a given functional. For $S \subset X$ we denoted toptdiam(S) as a *topological diameter* of S, where

$$\operatorname{toptdiam}(S) := \sup\Big\{A(x, y) : x, y \in S\Big\};$$

and, in connection with this, a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be *topological Cauchy sequence* iff

$$\lim_{n\to\infty}\Big(\operatorname{toptdiam}\{T^n(x) : k \geq n\}\Big) = 0;$$

and, a topological space X is said to be *orbitally complete* (or T-*orbitally complete*) iff every topological Cauchy sequence which is contained in $\mathcal{O}(x)$ for some $x \in X$ converges in X. A mapping $T : X \to X$ is said to be *orbitally continuous* iff $\xi, x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x))$.

A functional f mapping X into the reals is T-*orbitally lower semicontinuous* at $p \in X$ if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $f(p) \leq \liminf_{n\to\infty} f(x_n)$.

We are now in a position to formulate the following statement, which is a roofing for a great number of known results in the fixed point theory.

Theorem 1. *Let T be a mapping of a topological space $X := (X, A)$ into itself and let X be orbitally complete. Suppose that there exists a mapping $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ satisfying (Iφ) such that*

$$\text{(D)} \qquad A(Tx, Ty) \leq \varphi\Big(\operatorname{toptdiam}\{x, y, Tx, Ty, T^2x, T^2y, \ldots\}\Big)$$

and toptdiam $\mathcal{O}(x) \in \mathbb{R}_+^0$ *for all $x, y \in X$. If $x \mapsto$* toptdiam $\mathcal{O}(x)$ *or $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous or T orbitally continuous, and $A(a, b) = 0$ iff $a = b$, then T has a unique fixed point $\xi \in X$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to ξ for every $x \in X$.*

We begin the proof with a well known lemma which is fundamental in the following context.

Lemma 1 (Tasković [8]). *Let the mapping $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ have the property (Iφ). If the sequence (x_n) of nonnegative real numbers satisfies the condition*

$$x_{n+1} \leq \varphi(x_n), \quad n \in \mathbb{N},$$

then the sequence (x_n) tends to zero. The velocity of this convergence is not necessarily geometrical.

A brief first proof of this statement may be found in Tasković [8]. Other brief proofs for this we can see in Tasković [3], [4] and [5]. Also see Seneta [2].

Proof of Theorem 1. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ is a topological Cauchy sequence. It is easy to verify that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ satisfies the following inequality

$$\operatorname{toptdiam} \mathcal{O}(T^{n+1}x) \le \varphi(\operatorname{toptdiam} \mathcal{O}(T^n x))$$

for $n \in \mathbb{N}$, and hence applying Lemma 2 to the sequence $(\operatorname{toptdiam} \mathcal{O}(T^n x))$ we obtain $\lim_{n\to\infty} \operatorname{toptdiam} \mathcal{O}(T^n x) = 0$. This implies that $\{T^n x\}_{n\in\mathbb{N}}$ is a topological Cauchy sequence in X and, by T-orbital completeness, there is a $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \to \operatorname{toptdiam} \mathcal{O}(x)$ is T-orbitally lower semicontinuous at ξ,

$$A(\xi, T\xi) \le \operatorname{toptdiam} \mathcal{O}(\xi) \le \liminf(\operatorname{toptdiam} \mathcal{O}(T^n x)) = 0;$$

thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

On the other hand, if $x \mapsto A(x, Tx)$ is a T-orbitally lower semicontinuous at ξ we have

$$A(\xi, T\xi) \le \liminf A(T^n x, T^{n+1}x) \le \liminf(\operatorname{toptdiam} \mathcal{O}(T^n x)) = 0;$$

and thus again $T\xi = \xi$, i.e., we have again shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T. Also, if T is orbitally continuous the proof of previous fact is trivial.

We complete the proof by showing that T can have at most one fixed point: for, if $\xi \ne \eta$ were two fixed points, then

$$\begin{aligned} 0 &< \max\{A(\xi,\eta), A(\eta,\xi)\} = \max\{A(T\xi,T\eta), A(T\eta,T\xi)\} \le \\ &\le \varphi\Big(\operatorname{toptdiam}\{\xi,\eta,T\xi,T\eta,T^2\xi,T^2\eta,\ldots\}\Big) = \\ &= \varphi\Big(\max\{A(\xi,\xi), A(\eta,\eta), A(\xi,\eta), A(\eta,\xi)\}\Big) = \\ &= \varphi\Big(\max\{A(\xi,\eta), A(\eta,\xi)\}\Big) < \max\{A(\xi,\eta), A(\eta,\xi)\}, \end{aligned}$$

a contradiction. The proof is complete. □

As immediate consequence of the preceding Theorem 1, we obtain directly the following interesting cases of (D):

(1) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z\to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \le \psi(\operatorname{toptdiam}\{x, y, Tx, Ty\})$$

for all $x, y \in X$.

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha \in [0,1)$ such that for all $x, y \in X$ the following inequalities hold

$$A(Tx,Ty) \leq \alpha \,\mathrm{toptdiam}\{x,y,Tx,Ty\},$$

i.e., equivalently to

$$A(Tx,Ty) \leq \alpha \max\Big\{A(x,y), A(x,Tx), A(y,Ty), A(x,Ty), A(y,Tx)\Big\}.$$

(3) (*The condition of* $(m+k)$*-polygon*). There exists a constant $\alpha \in [0,1)$ such that for all $x, y \in X$ the following inequality holds in the form as

$$A(Tx,Ty) \leq \alpha \,\mathrm{toptdiam}\Big\{x,y,Tx,Ty,\ldots,T^m x, T^k y\Big\}$$

for arbitrary fixed integers $m, k \geq 0$. (This is a linear condition for diameter of finite number of points).

(4) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form $\limsup_{z \to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx,Ty) \leq \psi\Big(\mathrm{toptdiam}\{x,y,Tx,Ty,\ldots,T^m x, T^k y\}\Big)$$

for arbitrary fixed integers $m, k \geq 0$ and for all $x, y \in X$. (*This is a nonlinear condition for top.diameter of finite number of points*).

(5) There exists an increasing mapping for any coordinates of $f : (\mathbb{R}^0_+)^5 \to \mathbb{R}^0_+$ satisfying $\limsup_{z \to t+0} f(z,z,z,z,z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx,Ty) \leq f\Big(A(x,y), A(x,Tx), A(y,Ty), A(x,Ty), A(y,Tx)\Big)$$

for all $x, y \in X$.

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 1 in the following form.

Theorem 2. *Let T be a mapping of a topological space $X := (X, A)$ into itself and let X be orbitally complete. Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying* $(\mathrm{I}\varphi)$ *such that*

$$\mathrm{toptdiam}\{Tx, T^2x, \ldots\} \leq \varphi\Big(\mathrm{toptdiam}\{x, Tx, T^2x, \ldots\}\Big)$$

and $\mathrm{toptdiam}\,\mathcal{O}(x) \in \mathbb{R}^0_+$ *for every $x \in X$. If $x \mapsto$* $\mathrm{toptdiam}\,\mathcal{O}(x)$ *or $x \mapsto A(x,Tx)$ is T-orbitally lower semicontinuous or T is orbitally continuous, and $A(a,b) = 0$ iff $a = b$, then T has at least one fixed point in X.*

The proof of this localization statement is totally analogous with the preceding proof of Theorem 1. Thus the proof of this result we omit.

References

[1] M. Ohta and G. Nikaido: *Remarks on fixed point theorems in complete metric spaces*, Math. Japonica, **39** (1994), 287–290.

[2] E. Seneta: *Karamata's iteration theorem and normed regularly varying sequences in a historical light*, Publ. Inst. Math., Beograd, **48** (1990), 101–110.

[3] M. R. Tasković: *Some results in the fixed point theory-II*, Publ. Inst. Math., Beograd, **41** (1980), 249–258.

[4] M. R. Tasković: *Some theorems on fixed point and its applications*, Ann. Soc. Math. Polon. Ser. I, Comment. Math. Prace. Math., **24** (1984), 323–334.

[5] M. R. Tasković: *Some new principles in fixed point theory*, Math. Japonica, **35** (1990), 645–666.

[6] M. R. Tasković: *Geometric fixed point theorems on transversal spaces*, Math. Moravica, **8-2** (2004), 29–52.

[7] M. R. Tasković: *Theory of transversal point, spaces and forks*, Monographs of a new mathematical theory, VIZ-Beograd 2005, (in Serbian), 1054 pages. English summary: 1001–1022.

[8] M. R. Tasković: *A generalization of Banach's contraction principle*, Publ. Inst. Math. Beograd, **37** (1978), 179–191.

Milan R. Tasković
Faculty of Mathematics
P.O. Box 550
11000 Beograd
Serbia

Home address:
Milan R. Tasković
Nehruova 236
11070 Belgrade
Serbia
E-mail address: andreja@predrag.us