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Some Presic Type Generalizations of the Banach Contraction Principle

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Abstract. In this paper, we extend and generalize Presic Type theorems for a pair of maps and Jungck type maps.

1. Introduction and Preliminaries

In 1932 Banach [2] proved the following theorem:

Theorem 1.1 ([2])**.** *Let (X, d) be a complete metric space and let T be a mapping of X into X satisfying the inequality $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$, where $0 \leq \lambda < 1$. Then T has a unique fixed point in X.*

Since then, many generalizations of this principle have been made by several authors. Considering the convergence of certain sequences Presic [3] proved the following theorem.

Theorem 1.2 ([3])**.** *Let (X, d) be a complete metric space, k a positive integer and let T be a mapping of X^k into X, satisfying the following contractive type condition*

$$\begin{aligned} &d(T(x_1, x_2, x_3, \dots, x_k), T(x_2, x_3, x_4, \dots, x_k, x_{k+1})) \\ &\leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \cdots + q_k d(x_k, x_{k+1}) \end{aligned} \tag{1.1}$$

for every $x_1, x_2, x_3, x_4, \dots, x_k, x_{k+1} \in X$, where $q_1, q_2, \dots, q_k$ are non-negative constants such that $q_1 + q_2 + \ldots + q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, x, \dots, x) = x$.

Moreover, if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and if for all $n \in N$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

Ciric and Presic [1] generalized Theorem 1.2 as follows:

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Theorem 1.3. *Let (X, d) be a complete metric space, k a positive integer and let T be a mapping of X^k into X satisfying the following contractive type condition*

$$d(T(x_1, x_2, x_3, \ldots, x_k), T(x_2, x_3, x_4, \ldots, x_k, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1})/1 \leq i \leq k\} \tag{1.2}$$

for every $x_1, x_2, x_3, x_4, \ldots, x_k, x_{k+1} \in X$, where $0 < \lambda < 1$. Then there exists a point $x \in X$ such that $T(x, x, x, \ldots, x) = x$.

Moreover, if $x_1, x_2, x_3, \ldots, x_k$ are arbitrary points in X and if for all $n \in N$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$. If in addition, we suppose that on the diagonal $\Delta \subset X^k$, the condition

$$d(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v) \tag{1.3}$$

holds for all distinct $u, v \in X$, then x is the unique point in X with $T(x, x, \ldots, x) = x$.

Now in this paper we extend and generalize the above theorems for a pair of mappings and Jungck type mappings.

2. Main result

Theorem 2.1. *Let (X, d) be a complete metric space, k a positive integer and let S, T be mappings of X^{2k} into X satisfying the following contractive type conditions*

$$d(S(x_1, x_2, \ldots, x_{2k-1}, x_{2k}), T(x_2, x_3, \ldots, x_{2k}, x_{2k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : \ 1 \leq i \leq 2k\}, \tag{2.1}$$

for all $x_1, x_2, \ldots, x_{2k}, x_{2k+1} \in X$ and

$$d(T(y_1, y_2, \ldots, y_{2k-1}, y_{2k}), S(y_2, y_3, \ldots, y_{2k}, y_{2k+1})) \leq \lambda \max\{d(y_i, y_{i+1}) : \ 1 \leq i \leq 2k\}, \tag{2.2}$$

for all $y_1, y_2, \ldots, y_{2k}, y_{2k+1} \in X$, where $0 \leq \lambda < 1$.

Suppose $x_1, x_2, \ldots, x_{2k}$ are arbitrary points in X and for all $n \in N$ let

$$x_{2k+2n-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, \ldots, x_{2n+2k-2})$$

and

$$x_{2k+2n} = T(x_{2n}, x_{2n+1}, x_{2n+2}, \ldots, x_{2n+2k-1}).$$

Then the sequence $\{x_n\}$ is convergent to some $x \in X$ such that

$$S(x, x, \ldots, x) = x = T(x, x, \ldots, x). \tag{A}$$

In addition, if

(i) $2k\lambda < 1$, *or*
(ii) $d(S(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v)$,

for all distinct $u, v \in X$, then x is the unique point satisfying (A).

Proof. Let $\alpha_n = d(x_n, x_{n+1})$. We claim that $\alpha_n \le K\theta^n$, for all $n \in N$, where $\theta = \lambda^{1/2k}$ and $K = \max\{\alpha_1/\theta^1, \alpha_2/\theta^2, \dots, \alpha_{2k}/\theta^{2k}\}$. By selection of K we have $\alpha_n \le K\theta^n$ for $n = 1, 2, \dots, 2k$.

Now

$$\begin{aligned}
&\alpha_{2k+1} = d(x_{2k+1}, x_{2k+2})\\
&= d(S(x_1, x_2, \dots, x_{2k-1}, x_{2k}), T(x_2, x_3, \dots, x_{2k}, x_{2k+1}))\\
&\le \lambda \max\{d(x_i, x_{i+1}) :\ i = 1, 2, \dots, 2k\} \qquad \text{(by (2.1))}\\
&= \lambda \max\{\alpha_1, \alpha_2, \dots, \alpha_{2k-1}, \alpha_{2k}\}\\
&\le \lambda \max\{K\theta, K\theta^2, \dots, K\theta^{2k-1}, K\theta^{2k}\}\\
&= \lambda K\theta\\
&= K\theta^{2k+1} \qquad (\text{since } \theta = \lambda^{1/2k})
\end{aligned}$$

and so $\alpha_{2k+1} \le K\theta^{2k+1}$.

Similarly

$$\begin{aligned}
&\alpha_{2k+2} = d(x_{2k+2}, x_{2k+3})\\
&= d(T(x_2, x_3, \dots, x_{2k}, x_{2k+1}), S(x_3, x_4, \dots, x_{2k+1}, x_{2k+2}))\\
&\le \lambda \max\{d(x_i, x_{i+1}) :\ i = 2, 3, \dots, 2k+1\} \qquad \text{(by (2.2))}\\
&= \lambda \max\{\alpha_i / i = 2, 3, \dots, 2k+1\}\\
&\le \lambda \max\{K\theta^2, K\theta^3, \dots, K\theta^{2k+1}\}\\
&= \lambda K\theta^2\\
&= K\theta^{2k+2} \qquad (\text{since } \theta = \lambda^{1/2k})
\end{aligned}$$

and so $\alpha_{2k+2} \le K\theta^{2k+2}$. Hence our claim is true.

Now, by our claim, for any $n, p \in N$, we have

$$\begin{aligned}
&d(x_n, x_{n+p}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p})\\
&= \alpha_n + \alpha_{n+1} + \dots + \alpha_{n+p-1}\\
&\le K\theta^n + K\theta^{n+1} + \dots + K\theta^{n+p-1}\\
&\le K(\theta^n + \theta^{n+1} + \dots + \theta^{n+p-1} + \dots)\\
&= K\frac{\theta^n}{1-\theta} \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists a point $x \in X$ such that $x = \lim_{n\to\infty} x_n$. Then for any integer

n, using (2.1) and (2.2), we have

$$\begin{aligned}
& d(S(x,x,\ldots,x),x_{2n+2k-1}) = d(S(x,x,\ldots,x),S(x_{2n-1},x_{2n},\ldots,x_{2n+2k-2})) \\
& \le d(S(x,x,\ldots,x),T(x,x,\ldots,x,x_{2n-1})) \\
& \quad + d(T(x,x,\ldots,x,x_{2n-1}),S(x,x,\ldots,x_{2n-1},x_{2n})) \\
& \quad + d(S(x,x,\ldots,x,x_{2n-1},x_{2n}),T(x,x,\ldots,x,x_{2n-1},x_{2n},x_{2n+1})) \\
& \quad + d(T(x,x,\ldots,x,x_{2n},x_{2n+1}),S(x,x,\ldots,x,x_{2n},x_{2n+1},x_{2n+2})) + \ldots \\
& \quad + d(S(x,x,x_{2n-1},x_{2n},\ldots,x_{2n+2k-4}),T(x,x_{2n-1},x_{2n},\ldots,x_{2n+2k-3})) \\
& \quad + d(T(x,x_{2n-1},x_{2n},\ldots,x_{2n+2k-3}),S(x_{2n-1},x_{2n},\ldots,x_{2n+2k-2})) \\
& \le \lambda d(x,x_{2n-1}) + \lambda\max\{d(x,x_{2n-1}),d(x_{2n-1},x_{2n})\} \\
& \quad + \lambda\max\{d(x,x_{2n-1}),d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\} + \ldots \\
& \quad + \lambda\max\{d(x,x_{2n-1}),d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} \\
& \quad + \ldots + \lambda\max\{d(x,x_{2n-1}),d(x_{2n-1},x_{2n}),\ldots,d(x_{2n+2k-4},x_{2n+2k-3})\} \\
& \quad + \lambda\max\{d(x,x_{2n-1}),d(x_{2n-1},x_{2n}),\ldots,d(x_{2n+2k-3},x_{2n+2k-2})\}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$d(S(x,x,\ldots,x),x) \le 0$$

and so $S(x,x,\ldots,x) = x$.

From (2.1), we have

$$d(x,T(x,x,\ldots,x)) = d(S(x,x,\ldots,x),T(x,x,\ldots,x)) = 0$$

and so $T(x,x,\ldots,x) = x$.

To prove the uniqueness of x, we suppose that there exists a point $y \neq x$ in X such that

$$S(y,y,\ldots,y) = y = T(y,y,\ldots,y).$$

Suppose (i) holds so that $2k\lambda < 1$.

$$\begin{aligned}
& d(x,y) = d(S(x,x,\ldots,x),T(y,y,\ldots,y)) \\
& \le d(S(x,x,\ldots,x),T(x,x,\ldots,x,y)) + d(T(x,x,\ldots,x,y),S(x,x,\ldots,x,y,y)) \\
& \quad + d(S(x,x,\ldots,x,y,y),T(x,x,\ldots,x,y,y,y)) \\
& \quad + d(T(x,x,\ldots,x,y,y,y),S(x,x,\ldots,x,y,y,y,y)) \\
& \quad + \ldots + d(S(x,x,y,y,\ldots,y),T(x,y,y,\ldots,y)) \\
& \quad + d(T(x,y,y,\ldots,y),S(y,y,y,\ldots,y)) \\
& \quad + d(S(y,y,y,\ldots,y),T(y,y,y,\ldots,y)) \\
& \le \lambda d(x,y) + \lambda d(x,y) + \lambda d(x,y) + \ldots + \lambda d(x,y) + \lambda d(x,y) + 0 \\
& = 2K\lambda d(x,y) < d(x,y),
\end{aligned}$$

a contradiction. Therefore $y = x$. □

Suppose (ii) holds. Then

$$d(x,y) = d(S(x,x,\dots,x),T(y,y,\dots,y)) < d(x,y),$$

a contradiction and again $y = x$.

Corollary 2.1. *Let (X,d) be a complete metric space, k a positive integer and let S,T be mappings of X^{2k} into X satisfying*

$$\begin{aligned} &d(S(x_1,x_2,\dots,x_{2k-1},x_{2k}),T(x_2,x_3,\dots,x_{2k},x_{2k+1})) \\ &\le q_1 d(x_1,x_2) + q_2 d(x_2,x_3) + \ldots + q_{2k} d(x_{2k},x_{2k+1}), \end{aligned} \tag{2.3}$$

for all $x_1,x_2,x_3,\dots,x_{2k},x_{2k+1} \in X$ and

$$\begin{aligned} &d(T(y_1,y_2,\dots,y_{2k-1},y_{2k}),S(y_2,y_3,\dots,y_{2k},y_{2k+1})) \\ &\le q_1 d(y_1,y_2) + q_2 d(y_2,y_3) + \ldots + q_{2k} d(y_{2k},y_{2k+1}) \end{aligned} \tag{2.4}$$

for all $y_1,y_2,\dots,y_{2k},y_{2k+1} \in X$, where $q_1,q_2,\dots,q_{2k}$ are non-negative constants such that $q_1+q_2+\ldots+q_{2k} < 1$. Then there exists unique $x \in X$ such that

$$S(x,x,x,\dots,x) = x = T(x,x,x,\dots,x).$$

Proof. (2.3) and (2.4) imply the conditions (2.1) and (2.2) respectively with $\lambda = q_1+q_2+\ldots+q_{2k}$. Now from Theorem 2.1, there exists $x \in X$ such that

$$S(x,x,\dots,x) = x = T(x,x,\dots,x).$$

To prove the uniqueness of x, suppose there exists a point $y \neq x$ in X such that

$$S(y,y,\dots,y) = y = T(y,y,\dots,y).$$

Then

$$\begin{aligned} d(x,y) &= d(S(x,x,\dots,x),T(y,y,\dots,y)) \\ &\le d(S(x,x,\dots,x),T(x,x,\dots,x,y)) + d(T(x,x,\dots,x,y),S(x,x,\dots,x,y,y)) \\ &\quad + d(S(x,x,\dots,x,y,y),T(x,x,\dots,x,y,y,y)) \\ &\quad + d(T(x,x,\dots,x,y,y,y),S(x,x,\dots,x,y,y,y,y)) + \cdots \\ &\quad + d(S(x,x,y,y,\dots,y),T(x,y,y,\dots,y)) \\ &\quad + d(T(x,y,y,\dots,y),S(y,y,y,\dots,y)) \\ &\quad + d(S(y,y,y,\dots,y),T(y,y,y,\dots,y)) \\ &\le q_{2k} d(x,y) + q_{2k-1} d(x,y) + \ldots + q_2 d(x,y) + q_1 d(x,y) + 0 \\ &= (q_1+q_2+\ldots+q_{2k-1}+q_{2k}) d(x,y) < d(x,y), \end{aligned}$$

which is a contradiction. Therefore $y = x$. □

Definition 2.1. Let X be a non empty set, let T be a mapping of X^k into X and let f be a mapping of X into X. Then (f,T) is said to be weakly a k-compatible pair if $f(T(p,p,\dots,p)) = T(fp,fp,\dots,fp)$, whenever $p \in X$ is such that $fp = T(p,p,\dots,p)$.

Theorem 2.2. *Let (X, d) be a metric space, k a positive integer, let T be a mapping of X^k into X and let f be a mapping of X into X satisfying*

$$\begin{aligned} &d(T(x_1, x_2, x_3, \ldots, x_k), T(x_2, x_3, x_4, \ldots, x_k, x_{k+1})) \\ &\leq \lambda \max\{d(fx_i, fx_{i+1}) : \ 1 \leq i \leq k\}, \end{aligned} \tag{2.5}$$

for all $x_1, x_2, x_3, x_4, \ldots, x_k, x_{k+1} \in X$, where $0 < \lambda < 1$ and

$$d(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(fu, fv), \tag{2.6}$$

for all distinct $u, v \in X$. Suppose further that $T(X^k) \subseteq f(X)$, $f(X)$ is complete and (f, T) is a weakly k-compatible pair. Then there exists a unique point $z \in X$ such that

$$fz = z = T(z, z, \ldots, z).$$

Proof. Let $x_1, x_2, \ldots, x_k$ be arbitrary points in X and define

$$fx_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$$

for all $n \in N$. By proceeding as in [1], we can prove that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is complete, there exists a point $z \in f(X)$ such that $fx_n \longrightarrow z$. Hence there exists a point $p \in X$ such that $z = fp$.

Now consider

$$\begin{aligned} &d(fx_{n+k}, T(p, p, \ldots, p)) = d(T(p, p, \ldots, p), T(x_n, x_{n+1}, \ldots, x_{n+k-1})) \\ &\leq d(T(p, p, \ldots, p), T(p, p, \ldots, p, x_n)) \\ &\quad + d(T(p, p, \ldots, p, x_n), T(p, p, \ldots, p, x_n, x_{n+1})) \\ &\quad + d(T(p, p, \ldots, p, x_n, x_{n+1}), T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2})) \\ &\quad + d(T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2}), T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2}, x_{n+3})) \\ &\quad + \ldots + d(T(p, x_n, x_{n+1}, \ldots, x_{n+k-2}), T(x_n, x_{n+1}, \ldots, x_{n+k-1})) \\ &\leq \lambda d(fp, fx_n) + \lambda \max\{d(fp, fx_n), d(fx_n, fx_{n+1})\} \\ &\quad + \lambda \max\{d(fp, fx_n), d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2})\} \\ &\quad + \lambda \max\{d(fp, fx_n), d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), d(fx_{n+2}, fx_{n+3})\} \\ &\quad + \ldots + \lambda \max\{d(fp, fx_n), d(fx_n, fx_{n+1}), \ldots, d(fx_{n+k-2}, fx_{n+k-1})\}. \end{aligned}$$

Letting $n \longrightarrow \infty$ we get

$$d(fp, T(p, p, \ldots, p)) \leq 0$$

so that $fp = T(p, p, \ldots, p)$.

Since (f, T) is weakly k-compatible we have

$$f(T(p, p, \ldots, p)) = T(fp, fp, \ldots, fp)$$

and so

$$f^2p = f(fp) = f(T(p, p, \ldots, p)) = T(fp, fp, \ldots, fp).$$

Thus

$$fz = T(z, z, \ldots, z). \tag{i}$$

We now have

$$d(f^2p, fp) = d(T(fp, fp, \dots, fp), T(p, p, \dots, p)) < d(f^2p, fp),$$

which is a contradiction. Therefore $f^2p = fp$ so that $fz = z$.

From (i), we now have

$$z = fz = T(z, z, \dots, z). \tag{ii}$$

To prove uniqueness, suppose that there exists a point $z^1 \neq z$ in X such that

$$z^1 = fz^1 = T(z^1, z^1, \dots, z^1).$$

Then

$$\begin{aligned} d(z, z^1) &= d(T(z, z, \dots, z), (T(z^1, z^1, \dots, z^1)) \\ &< d(fz, fz^1) \qquad \text{from (2.6)} \\ &= d(z, z^1), \end{aligned}$$

which is a contradiction. Therefore $z = z^1$ proving that z is the unique point satisfying (ii). □

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