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Perturbation of Farthest Points in Weakly Compact Sets

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Abstract. If f is a real valued weakly lower semi-continuous function on a Banach space X and C a weakly compact subset of X, we show that the set of $x \in X$ such that $z \mapsto \|x-z\| - f(z)$ attains its supremum on C is dense in X. We also construct a counter example showing that the set of $x \in X$ such that $z \mapsto \|x - z\| + \|z\|$ attains its supremum on C is not always dense in X.

1. Introduction

Throughout this paper, X denotes a real Banach space, B_X its closed unit ball, X^* the Banach space of all continuous linear functionals on X, C a bounded set of X and $f : X \to \mathbb{R}$ a function which is bounded below on C. We study the following sets

$$D(C, f) = \big\{x \in X; \exists z \in C, r(x) = \|x - z\| - f(z)\big\},$$

where by definition r is the map from X to $\mathbb{R}$ given by the formula

$$r(x) = \sup\big\{\|x - z\| - f(z), z \in C\big\}.$$

The map r depends on f and should be written r_f, but since there will be no ambiguity, we simply write $r = r_f$. We remark that r is 1-Lipschitz and convex as a supremum of such functions and that by replacing f by $f + a$ where a is a constant, we can suppose that $f \geqslant 0$. When $f = 0$, the set $D(C, 0)$ is geometrically the set of points of X which admit a farthest point in the set C and $r(x)$ is the farthest distance from x to C, i.e. $r(x)$ is the smallest radius of the balls centered in x that contain C. Here, the function f is a perturbation, we will show that under suitable hypothesis of regularity on f, some results known on the set $D(C, 0)$ can be generalized. To be more precise, we will be interested in the generic existence of points in $D(C, f)$. For farthest points, the problem was first studied by Edelstein in [2] for uniformly convex spaces, assuming the set C is bounded and norm closed and then generalized by Asplund in [1] for reflexive locally uniformly convex spaces. Then Lau in [4] showed that when C is weakly compact (without any geometric hypothesis on X), the set of farthest points is dense and he

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also showed that this result implies Asplund's theorem. Here we will give a generalization of Lau's theorem (see also the paper [5] which deals with Euclidean spaces, and [3] for the case of p-normed spaces): when f is weakly lower semi-continuous and C weakly compact, the set $D(C, f)$ contains a G_δ dense subset of X. We then take some particular f to see what happens when we study the set of points $x \in X$ such that $z \mapsto \|z - x\| - \|z\|$ (resp. $z \mapsto \|z - x\| + \|z\|$) attain their supremum on C.

2. DENSITY OF THE SET $D(C, f)$

We start this section by defining the sub-differential of the map r (this definition stays unchanged for any convex map).

Definition 2.1. *The sub-differential of r is the set*

$$\partial r(x) = \{x^* \in X^*; \forall y \in X, \langle x^*, y - x\rangle \leqslant r(y) - r(x)\}.$$

Since r is 1-Lipschitz, $\partial r(x)$ is contained in the closed unit ball of the dual. We can now state our positive theorem which follows the ideas of Lau's proof.

Theorem 2.1. *Suppose that C is a weakly compact subset of X and that f is weakly lower semi-continuous for the weak topology on X, then the set $D(C, f)$ contains a G_δ dense subset of X.*

In order to prove the theorem, we will use the following lemma:

Lemma 2.1. *Let $G = \{x \in X; \forall x^* \in \partial r(x), \sup\{\langle x^*, x - z\rangle - f(z), z \in C\} = r(x)\}$. Then G is a G_δ dense subset of X.*

Proof. Write $X \setminus G = \bigcup_{n=1}^{\infty} F_n$ with

$$F_n = \left\{x \in X; \exists x^* \in \partial r(x), \sup\{\langle x^*, x - z\rangle - f(z), z \in C\} \leqslant r(x) - \frac{1}{n}\right\}.$$

By the Baire category theorem, it is enough to show that for fixed $n \geqslant 1$, F_n is closed and nowhere dense.

– Let us first show that F_n is a closed subset of X: let (x_k) be a sequence in F_n converging to $x \in X$. By the definition of F_n, there exists $x_k^* \in \partial r(x_k)$ such that

$$\forall z \in C, \forall k \geqslant 1, \langle x_k^*, x_k - z\rangle - f(z) \leqslant r(x_k) - \frac{1}{n}.$$

Since B_{X^*} is compact for $\sigma(X^*, X)$, we can choose $x^* \in \bigcap_p \overline{\{x_k^*, k \geqslant p\}}^{\sigma(X^*,X)}$, then we get for $z \in C$:

$$\begin{aligned}
|\langle x_k^*, x_k - z\rangle - \langle x^*, x - z\rangle| &\leqslant |\langle x_k^*, x_k - z\rangle - \langle x_k^*, x - z\rangle| \\
&\quad + |\langle x_k^*, x - z\rangle - \langle x^*, x - z\rangle| \\
&\leqslant \|x_k^*\|\|x_k - x\| + |\langle x_k^*, x - z\rangle - \langle x^*, x - z\rangle| \\
&\leqslant \|x_k - x\| + |\langle x_k^*, x - z\rangle - \langle x^*, x - z\rangle|.
\end{aligned}$$

Now for each fixed $z \in C$, there exists a subsequence $(x^*_{k_q})$ such that $\langle x^*_{k_q}, x - z\rangle$ converges, and because $x^* \in \bigcap_p \overline{\{x^*_k, k \geqslant p\}}^{\sigma(X^*,X)}$, this limit is $\langle x^*, x - z\rangle$. By continuity of r, we obtain for each $z \in C$

$$\langle x^*, x - z\rangle - f(z) \leqslant r(x) - \frac{1}{n},$$

and hence

$$\sup\{\langle x^*, x - z\rangle - f(z), z \in C\} \leqslant r(x) - \frac{1}{n}.$$

To conclude that $x \in F_n$, it is enough to show that $x^* \in \partial r(x)$. Indeed, since $x^*_k \in \partial r(x_k)$, we have

$$\forall y \in X, \langle x^*_k, y - x_k\rangle \leqslant r(y) - r(x_k)$$

so by the same argument as before, we get at the limit: $x^* \in \partial r(x)$.

– Now, let us show that each F_n is nowhere dense. Suppose it is false, then one can find $y_0 \in X$ and $r > 0$ such that $\overline{B}(y_o, r) \subset F_n$. Let $\alpha = \sup\{\|z\|, z \in C\}$, $\lambda = \frac{r}{\alpha + \|y_0\|}$ and $\varepsilon = \frac{\lambda}{n(1+\lambda)}$. By the definition of $r(y_0)$, there exists $z_0 \in C$ such that

$$r(y_0) - \varepsilon < \|y_0 - z_0\| - f(z_0) \leqslant r(y_0).$$

Finally, put $x_0 = y_0 + \lambda(y_0 - z_0)$. With the choice of λ, we have $x_0 \in \overline{B}(y_0, r) \subset F_n$. Now, we estimate $r(y_0) - r(x_0)$:

$$r(y_0) - r(x_0) < \varepsilon + \|y_0 - z_0\| - f(z_0) - r(x_0).$$

But,

$$x_0 = y_0 + \lambda(y_0 - z_0) \Longrightarrow x_0 - z_0 = (1 + \lambda)(y_0 - z_0).$$

Hence

$$\begin{aligned} r(y_0) - r(x_0) &< \varepsilon + \frac{1}{1+\lambda}\|x_0 - z_0\| - f(z_0) - r(x_0) \\ &= \varepsilon + \frac{1}{1+\lambda}(\|x_0 - z_0\| - f(z_0)) + \left(\frac{1}{1+\lambda} - 1\right) f(z_0) - r(x_0) \\ &\leqslant \varepsilon + \frac{1}{1+\lambda} r(x_0) - \frac{\lambda}{1+\lambda} f(z_0) - r(x_0) \\ &= \varepsilon - \frac{\lambda}{1+\lambda} r(x_0) - \frac{\lambda}{1+\lambda} f(z_0). \end{aligned}$$

Since $x_0 \in F_n$, there exists $x^* \in \partial r(x_0)$ such that

$$r(x_0) \geqslant \sup\{\langle x^*, x_0 - z\rangle - f(z), z \in C\} + \frac{1}{n} \geqslant \langle x^*, x_0 - z_0\rangle - f(z_0) + \frac{1}{n},$$

which gives, combined with the last estimation:

$$r(y_0) - r(x_0) < \varepsilon - \frac{\lambda}{1+\lambda}\langle x^*, x_0 - z_0\rangle - \varepsilon = \langle x^*, y_0 - x_0\rangle,$$

which contradicts $x^* \in \partial r(x_0)$. $\square$

Here, we have just used the fact that C is bounded. The hypothesis of weak compactness of C and of weak lower semi-continuity of f allow us to finish the proof of the theorem as follows.

Proof. It is enough to see that $G \subset D(C, f)$. Consider $x \in G$ and $x^* \in \partial r(x)$, so

$$\sup\{\langle x^*, x - z\rangle - f(z), z \in C\} = r(x).$$

Since f is weakly lower semi-continuous and that $z \mapsto \langle x^*, x - z\rangle$ is weakly continuous, then $z \mapsto \langle x^*, x - z\rangle - f(z)$ is weakly upper semi-continuous on the weakly compact set C, and attains its supremum at a point z_0. We get:

$$r(x) \leqslant \|x^*\|\|x - z_0\| - f(z_0) \leqslant r(x)$$

because $\|x^*\| \leqslant 1$ and hence $r(x) = \|x - z_0\| - f(z_0)$. □

Since $z \mapsto \|z\|$ is weakly lower semi-continuous, we obtain

Corollary 2.1. *If C is weakly compact, the set of $x \in X$ such that $z \mapsto \|x - z\| - \|z\|$ attains its supremum on C is dense in X.*

3. COUNTER EXAMPLES AND REMARKS

It is natural to ask ourselves if we can drop the hypothesis of weak lower semi-continuity in Theorem 2.1. The answer is no: more precisely, we construct the following counter example

Example 3.1. *If (K, d) is an infinite compact metric space and if $X = C(K)$ is the space of real continuous functions on K equiped with its usual norm, there exists a weakly compact subset C of X and a function f weakly upper semi-continuous on X such that $D(C, f)$ is not dense in X.*

Indeed, take $f(z) = (1 - \|z\|)^+ = \max(0, 1 - \|z\|)$ and consider a decreasing sequence $(U_n)_{n\geqslant 1}$ of open subsets of K such that $\bigcap_{n\geqslant 1} U_n = \emptyset$ (fix $y \in K$ which is not an isolated point in K, then a possible choice is $U_n = \{x \in K \setminus \{y\}; d(x, y) < \frac{1}{n}\}$), let us also fix $t_n \in U_n$ and put

$$x_n(t) = \frac{d(t, U_n^c)}{d(t, t_n) + d(t, U_n^c)} \quad (t \in K, n \geqslant 1).$$

By construction of U_n, we have $\|x_n\| = 1$ and $(x_n)_{n\geqslant 1}$ converges pointwise to 0 which implies that $(x_n)_{n\geqslant 1}$ converges weakly to 0 as easily seen using the Riesz representation theorem and the Lebesgue's dominated convergence theorem. Put

$$C = \left\{\left(1 - \frac{1}{n}\right) x_n, n \geqslant 1\right\} = \{0\} \cup \left\{\left(1 - \frac{1}{n}\right) x_n, n \geqslant 2\right\}$$

which is weakly compact as the union of a convergent sequence and its limit. Note that C is contained in B_X and hence $f(z) = 1 - \|z\|$, we are left to find the supremum of the function f_x ($x \in X$ fixed) defined for $z \in C$ by $f_x(z) = \|x - z\| + \|z\|$. We will show that for $x \in \overline{B}(\mathbf{2}, 1)$ (where $\mathbf{2}$ denotes the function identically equal to 2), f_x never attains its supremum and as a consequence $D(C,f)$ is not dense. Since for $t \in K$, $x(t) \geqslant 1$, we get for $z \in C$

$$\|x-z\| = \sup |x(t)-z(t)| = \sup(x(t)-z(t)) \leqslant \sup x(t) = \|x\|$$

and on the other hand $\|z\| < 1$ gives $f_x(z) < \|x\| + 1$. To finish, the last thing we have to see is that $\sup f_x \geqslant \|x\| + 1$. Fix t_0 such that $\|x\| = |x(t_0)|$, then

$$\sup f_x \geqslant f_x\left(\left(1-\frac{1}{n}\right)x_n\right) \geqslant \left|x(t_0) - \left(1-\frac{1}{n}\right)x_n(t_0)\right| + \left(1-\frac{1}{n}\right).$$

The conclusion follows because $(x_n)_{n\geqslant 1}$ converges pointwise to 0.

Remark 3.1. – *This last example also shows that the set of $x \in X$ such that $z \mapsto \|z-x\| + \|z\|$ attains its supremum on C is not always dense in X. Recall that according to Corollary 2.1, the set of $x \in X$ such that $z \mapsto \|z-x\| - \|z\|$ attains its supremum on C is always dense in X.*
– There exists spaces, for example $l^1(\mathbb{N})$, or more generally any Banach space with the Schur's property where we can't construct any counter examples of the above type because the weakly and strongly compact sets coincide.

– However if $C = B_X$ and X is reflexive (to ensure the weak compactness of C). The set of x such that f_x (defined by $f_x(z) = \|x-z\| + \|z\|$) attains its supremum on C is dense. To show this, we use the following proposition.

Proposition 3.1. *Let f be a continuous convex function on X, C a weakly compact subset of X and $\varepsilon(C)$ the set of extremal points of C, then $\sup_C f = \sup_{\varepsilon(C)} f$.*

Proof. We have obviously, $\sup_{\varepsilon(C)} f \leqslant \sup_C f$. Suppose the reverse inequality is false and introduce t such that

$$\sup_{\varepsilon(C)} f < t < \sup_C f.$$

Then, we have $\varepsilon(C) \subset C_0 := \{f \leqslant t\}$. Since f is continuous convex , C_0 is a closed convex set, the Krein-Milman's theorem says that $\overline{\text{conv}}^{\|\cdot\|}(\varepsilon(C)) = C$, hence $C \subset C_0$. Now, since $\sup_C f > t$, one can find $x \in C$ such that $f(x) > t$ which contradicts $x \in C_0$. $\square$

This implies the last remark, indeed $\varepsilon(C)$ is of course contained in the unit sphere. Using the previous fact two times, we see that

$$\sup_{z\in C} f_x(z) = \sup_{z\in\varepsilon(C)} f_x(z) = 1 + \sup_{z\in\varepsilon(C)} \|x-z\| = 1 + \sup_{z\in C} \|x-z\|$$

which gives the conclusion with the main theorem (with the pertubation $f = 0$).

Remark 3.2. *To finish, we would like to mention that the map* $f \mapsto D(C,f)$ *has no good properties. Let us take* $X = \mathbb{R}$, $C = [0,1]$ *and put for* $z \in \mathbb{R}$, $f_k(z) = \frac{\mathbf{1}_{\{0,1\}}(z)}{k}$ *where* $\mathbf{1}_{\{0,1\}}$ *denotes the characteristic function of the pair* $\{0,1\}$ *which is equal to 1 if* $z = 0$ *or* $z = 1$ *and* 0 *otherwise. It is obvious that* $(f_k)_{k \geqslant 1}$ *converges uniformly to* 0 *(*$D(C,0) = X$*) and yet, all the* $D(C,f_k)$ *are empty.*

Indeed, let $x \in \mathbb{R}$ and suppose that $x \geqslant \frac{1}{2}$. For $z \in [0,1]$, $|x-z|$ is maximal when $z = 0$ and is equal to x. Hence

$$\sup\{|x-z| - f_k(z), z \in [0,1]\} \leqslant x.$$

On the other hand, taking a sequence $(z_n) \subset]0,1[$ converging to 0, we get the reverse inequality. If we had a z which attains the supremum, we should have

$$f_k(z) = |x-z| - x \leqslant x - x = 0,$$

which implies that $z \in]0,1[$. This gives us $|z-x| = x$ with $z \in]0,1[$, which contradicts $|x-z| < x$. For $x \leqslant \frac{1}{2}$, we proceed the same way with the point $z = 1$.

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