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Transversal Spring Spaces, the Equation $x = T(x, \dots, x)$ and Applications

Milan R. Tasković

Abstract. This paper continues the study of the transversal spaces. In this sense we formulate a new structure of spaces which we call it transversal (upper, lower, or middle) spring spaces. Also, we consider problems of the fixed point theory on transversal spring spaces. In connection with this, we give some solutions for the equation $x = T(x, \dots, x)$. This paper presents an extended asymptotic fixed point theory.

1. Transversal spring spaces

The upper spring transversal spaces. In connection with the former facts, we shall introduce the concept of a transversal spring upper space. In this sense, the function $A : X \times X \to \mathbb{R}^0_+ := [0, +\infty)$ is called an **upper spring transverse** on a nonempty set X (or *upper spring transversal*) iff: $A(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.

An **upper spring transversal space** $X := (X, A)$ is a nonempty set X together with a given upper spring transverse A on X.

Otherwise, the function A is called a **semiupper spring transverse** on a nonempty set X iff: $A(x, y) = 0$ implies $x = y$ for all $x, y \in X$. A **semiupper spring transversal space** $X := (X, A)$ is a nonempty set X together with a given semiupper spring transverse A on X.

Let $X := (X, A)$ be an upper spring transversal space, where $T : X \to X$, and $A : X \times X \to \mathbb{R}^0_+ := [0, +\infty)$ is a given functional. For $S \subset X$ we denoted $\operatorname{trstdiam}(S)$ as a *transversal spring diameter* of S, in the sense that

$$\operatorname{trstdiam}(S) := \sup \Big\{ A(x, y) : x, y \in S \Big\},$$

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where $S \subset Y$ implies $\operatorname{trstdiam}(S) \leqslant \operatorname{trstdiam}(Y)$.

Elements of an upper spring transversal space will usually be called *points*. Given an upper spring transversal space $X := (X, A)$, and a point $z \in X$, the **open ball** of center z and radius $r > 0$ is the set

$$A(B(z,r)) = \{x \in X : A(z,x) < r\}.$$

The *upper spring convergence* $x_n \to x$ as $n \to \infty$ in the upper spring transversal space $X := (X, A)$ means that the following fact holds that

$$A(x_n, x) \to 0 \quad \text{as} \quad n \to \infty,$$

or equivalently, for every $\varepsilon > 0$ there exist an integer n_0 such that the relation $n \geqslant n_0$ implies $A(x_n, x) < \varepsilon$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in the upper spring transversal space $X := (X, A)$ is called **upper spring transversal sequence** (or *upper spring Cauchy sequence*) iff: for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that

$$A(x_n, x_m) < \varepsilon \quad \text{for all} \quad n, m \geqslant n_0.$$

Let X be an upper spring transversal space and $T : X \to X$. We notice, from Tasković [28], that a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be upper spring transversal sequence if and only if

$$\lim_{n\to\infty} \Big(\operatorname{trstdiam}\{T^k(x) : k \geqslant n\} \Big) = 0.$$

In this sense, an upper spring transversal space is called **upper spring complete** iff every upper spring transversal sequence upper spring converges. Also, a space $X := (X, A)$ is said to be **upper spring orbitally complete** (or *upper spring T-orbitally complete*) iff every upper spring transversal sequence which contained in $\mathcal{O}(x)$ for some $x \in X$ upper spring converges in X.

In connection with the preceding, the set $\mathcal{O}(x, \infty) := \{x, Tx, T^2x, \dots\}$ for $x \in X$ is called the **orbit** of x. A function f mapping X into reals is a **f-orbitally lower semicontinuous** at the point p iff for all sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to p$ $(n \to \infty)$ it follows that $f(p) \leq \liminf_{n\to\infty} f(x_n)$. A mapping $T : X \to X$ is said to be orbitally continuous if ξ, $x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x, \infty)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x, \infty))$.

We are now in a position to formulate the following statement, which is roofing for a great number of known results on metric spaces and general in the fixed point theory.

Theorem 1. *Let T be a mapping of an upper spring transversal space $X := (X, A)$ into itself and let X be upper spring T-orbitally complete. Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying*

$$(\mathrm{I}\varphi) \qquad \Big(\forall t \in \mathbb{R}_+ := (0, +\infty)\Big) \left(\varphi(t) < t \text{ and } \limsup_{z\to t+0} \varphi(z) < t\right)$$

such that

$$\text{(D)} \qquad A\Big(Tx, Ty\Big) \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{x, y, Tx, Ty, T^2x, T^2y, \dots\Big\}\Big)$$

and $\operatorname{trstdiam}\mathcal{O}(x) \in \mathbb{R}^0_+$ *for all* $x, y \in X$. *If* $x \mapsto \operatorname{trstdiam}\mathcal{O}(x)$ *or* $x \mapsto A(x, Tx)$ *is* T*-orbitally lower semicontinuous, then* T *has a unique fixed point* $\zeta \in X$ *and* $\{T^n(x)\}_{n\in\mathbb{N}}$ *converges to* ζ *for every* $x \in X$.

Proof. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ is an upper transversal spring Cauchy sequence. It is easy to verify that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ satisfies the following inequality

$$\operatorname{trstdiam}\mathcal{O}(T^{n+1}x) \leqslant \varphi(\operatorname{trstdiam}\mathcal{O}(T^n x))$$

for $n \in \mathbb{N}$, and hence applying Lemma 1 by Tasković [32] to the sequence $(\operatorname{trstdiam}\mathcal{O}(T^n x))$ we obtain that $\lim_{n\to\infty} \operatorname{trstdiam}\mathcal{O}(T^n x) = 0$. This implies that $\{T^n x\}_{n\in\mathbb{N}}$ is an upper transversal spring Cauchy sequence in X and, by upper spring T-orbital completeness, there is a $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \longmapsto \operatorname{trstdiam}\mathcal{O}(x)$ is T-orbitally lower semicontinuous at ξ,

$$A(\xi, T\xi) \leqslant \operatorname{trstdiam}\mathcal{O}(\xi) \leqslant \liminf(\operatorname{trstdiam}\mathcal{O}(T^n x)) = 0;$$

thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T. On the other hand, if $x \mapsto A(x, Tx)$ is a T-orbitally lower semicontinuous at ξ we have

$$A(\xi, T\xi) \leqslant \lim_{n\to\infty} A(T^n x, T^{n+1}x) \leqslant \lim_{n\to\infty}(\operatorname{trstdiam}\mathcal{O}(T^n x)) = 0;$$

and thus again $T\xi = \xi$, i.e., we have again shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ upper converges to a fixed point of T.

We complete the proof by showing that T can have at most one fixed point: for, if $\xi \neq \eta$ were two fixed points, then

$$\begin{aligned} 0 < \max\{A(\xi,\eta), A(\eta,\xi)\} &= \max\{A(T\xi, T\eta), A(T\eta, T\xi)\} \leqslant \\ &\leqslant \varphi\Big(\operatorname{trstdiam}\{\xi, \eta, T\xi, T\eta, T^2\xi, T^2\eta, \dots\}\Big) = \\ &= \varphi\Big(\max\{A(\xi,\xi), A(\eta,\eta), A(\xi,\eta), A(\eta,\xi)\}\Big) = \\ &= \varphi\Big(\max\{A(\xi,\eta), A(\eta,\xi)\}\Big) < \max\{A(\xi,\eta), A(\eta,\xi)\}, \end{aligned}$$

a contradiction. The proof is complete.

As immediate consequences of the preceding Theorem 1, we obtain directly the following interesting cases of (D):

(1) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z\to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant \psi(\operatorname{trstdiam}\{x, y, Tx, Ty\}) \quad \text{for all} \quad x, y \in X.$$

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha \in [0, 1)$ such that for all $x, y \in X$ the following inequality holds

$$A(Tx, Ty) \leqslant \alpha \operatorname{trstdiam}\{x, y, Tx, Ty\},$$

i.e., equivalently to

$$A(Tx, Ty) \leqslant \alpha \max\Big\{A(x, y), A(x, Tx), A(y, Ty), A(x, Ty), A(y, Tx)\Big\}.$$

(3) (*The condition of* $(m + k)$*-polygon*). There exists a constant $\alpha \in [0, 1)$ such that for all $x, y \in X$ the following inequality holds in the form as

$$A(Tx, Ty) \leqslant \alpha \operatorname{trstdiam}\Big\{x, y, Tx, Ty, \ldots, T^m x, T^k y\Big\}$$

for arbitrary fixed integers $m, k \geqslant 0$. (*This is a linear condition for trs.diameter of finite number of points*).

(4) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z \to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant \psi\Big(\operatorname{trstdiam}\{x, y, Tx, Ty, \ldots, T^m x, T^k y\}\Big)$$

for arbitrary fixed integers $m, k \geqslant 0$ and for all $x, y \in X$. (*This is a nonlinear condition for trs.diameter of finite number of points*).

(5) There exists an increasing mapping for any coordinates of $f : (\mathbb{R}^0_+)^5 \to \mathbb{R}^0_+$ satisfying the following condition in form as $\limsup_{z \to t+0} f(z, z, z, z, z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant f\Big(A(x, y), A(x, Tx), A(y, Ty), A(x, Ty), A(y, Tx)\Big) \quad \text{for all} \quad x, y \in X.$$

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 1 in the following form.

Theorem 2. *Let T be a mapping of an upper spring transversal space $X := (X, A)$ into itself and let X be upper spring T-orbitally complete. Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying* $(\mathrm{I}\varphi)$ *such that*

$$\operatorname{trstdiam}\{Tx, T^2x, \ldots\} \leqslant \varphi\Big(\operatorname{trstdiam}\{x, Tx, T^2x, \ldots\}\Big)$$

and $\operatorname{trstdiam} \mathcal{O}(x) \in \mathbb{R}^0_+$ *for every $x \in X$. If $x \mapsto$* $\operatorname{trstdiam} \mathcal{O}(x)$ *or $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous, then T has at least one fixed point in X.*

The proof of this localization statement is totally analogous with the preceding proof of Theorem 1. Thus the proof of this result we omit.

Asymptotic contractions on upper spring transversal spaces. Let X be a nonempty set, $T : X \to X$, and let $A : X \times X \to \mathbb{R}^0_+$ be a given function. In 1986 we investigated the concept of upper spring TCS-convergence in a space X, i.e., an upper spring transversal space $X := (X, A)$ satisfies the **condition of upper spring TCS-convergence** iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to 0$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n \in \mathbb{N}}$ has a convergent subsequence in X.

Theorem 3. *Let T be a mapping of upper spring transversal space $X := (X, A)$ into itself, where X satisfies the condition of upper spring TCS-convergence. Suppose that for all $x, y \in X$ there exist a sequence of nonnegative real functions $\{\alpha_n(x, y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, y) \to 0$ $(n \to \infty)$ and positive integer $m(x, y)$ such that*

$$\text{(B)} \qquad A(T^n(x), T^n(y)) \leqslant \alpha_n(x, y) \quad \textit{for all} \quad n \geqslant m(x, y),$$

where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x, T(x))$ is a T-orbitally lower semicontinuous function, then T has a unique fixed point $\xi \in X$ and $T^n(x) \to \xi$ $(n \to \infty)$ for each $x \in X$.

Proof. For $y = T(x)$ from (B) we have that $A(T^n x, T^{n+1}x) \leqslant \alpha_n(x, Tx)$ for all $n \geqslant m(x, Tx)$), and thus we obtain that $A(T^n x, T^{n+1}x) \to 0$ $(n \to \infty)$. This implies (from upper spring TCS-convergence) that the sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{T^{n(i)}(x)\}_{i\in\mathbb{N}}$ with the limit point $\xi \in X$. Since $x \mapsto A(x, T(x))$ is T-orbitally lower semicontinuous, we get

$$A(\xi, T(\xi)) \leqslant \liminf_{i\to\infty} A(T^{n(i)}x, T^{n(i)+1}x) = \liminf_{n\to\infty} A(T^n x, T^{n+1}x) = 0,$$

which implies that $A(\xi, T(\xi)) = 0$, i.e., $\xi = T(\xi)$. We complete the proof by showing that T can have at most one fixed point. Indeed, if we suppose that $\xi \neq \eta$ were two fixed points, then from (B) we have

$$0 < A(\xi, \eta) = A(T^n(\xi), T^n(\eta)) \leqslant \alpha_n(\xi, \eta) \quad \text{for every} \quad n \geqslant m(\xi, \eta);$$

taking limits as $n \to \infty$ we obtain a contradiction. Thus we obtain that $\xi = \eta$, i.e., T has a unique fixed point $\xi \in X$. The proof is complete.

Remark. We notice that Theorem 3 is a generalization of *Caccioppoli's theorem* as well as many others on upper spring transversal spaces.

Note that, from the preceding proof of Theorem 3, we can give the following local form of this statement.

Theorem 4. (Localization of (B)). *Let T be a mapping of upper spring transversal space $X := (X, A)$ into itself, where X satisfies the condition of upper spring TCS-convergence. Suppose that for each $x \in X$ there exist a sequence of nonnegative real functions $\{\alpha_n(x, Tx)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, Tx) \to 0$ $(n \to \infty)$ and positive integer $m(x)$ such that*

$$A(T^n(x), T^{n+1}(x)) \leqslant \alpha_n(x, Tx) \quad \textit{for all} \quad n \geqslant m(x),$$

where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x, Tx)$ is a T-orbitally lower semicontinuous function, then T has at least one fixed point in X.

Fundamental fact. We notice, in connection with the preceding, that Theorems 3 and 4 de facto hold and for upper and semiupper topological spaces in suitable context and transcription.

The lower spring transversal spaces. In connection with the preceding, we shall introduce the concept of a lower spring transversal space. In this sense, the function $A : X \times X \to [0,+\infty] := \mathbb{R}^0_+ \cup \{+\infty\}$ is called a **lower spring transverse** on a nonempty set X (or *lower spring transversal*) iff: $A(x,y) = +\infty$ if and only if $x = y$ for all $x, y \in X$.

A **lower spring transversal space** $X := (X, A)$ is a nonempty set X together with a given lower spring transverse A on X.

Otherwise, the function A is called a **semilower spring transverse** on a nonempty set X iff: $A(x,y) = +\infty$ implies $x = y$ for all $x, y \in X$. A **semilower spring transversal space** $X := (X, A)$ is a nonempty set X together with a given semilower spring transverse A on X.

For any nonempty set S in the lower spring transversal space $X := (X, A)$ the **trs.diameter** of S is defined as

$$\operatorname{trstdiam}(S) := \inf \big\{A(x,y) : x, y \in S\big\};$$

where $Y \subset B$ implies $\operatorname{trstdiam}(B) \leqslant \operatorname{trstdiam}(Y)$. The relation $\operatorname{trstdiam}(S) = +\infty$ holds if and only if S is a one point set.

Elements of a lower spring transversal space will usually be called *points*. Given a lower spring transversal space $X := (X, A)$, and a point $z \in X$, the **open ball** of center z and radius $r > 0$ is the set

$$A(B(z,r)) = \big\{x \in X : A(z,x) > r\big\}.$$

On the other hand, from Tasković [28], the *lower spring convergence* $x_n \to x$ as $n \to \infty$ in the lower spring transversal space $X := (X, A)$ means that

$$A(x_n, x) \to +\infty \quad \text{as} \quad n \to \infty,$$

or equivalently, for every $\varepsilon > 0$ there exist an integer n_0 such that the relation $n \geqslant n_0$ implies $A(x_n, x) > \varepsilon$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in the lower spring transversal space $X := (X, A)$ is called **lower spring transversal sequence** (or *lower spring Cauchy sequence*) iff for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that

$$A(x_n, x_m) > \varepsilon \quad \text{for all} \quad n, m \geqslant n_0.$$

Let $X := (X, A)$ be a lower spring transversal space and $T : X \to X$. We notice, from Tasković [28], that a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be lower spring transversal sequence if and only if

$$\lim_{n\to\infty} \Big(\operatorname{trstdiam}\{T^k(x) : k \geqslant n\}\Big) = +\infty.$$

In this sense, a lower spring transversal space is called **lower spring complete** iff every lower spring transversal sequence lower spring converges.

Also, a space (X, ρ) is said to be **lower spring orbitally complete** (or *lower spring T-orbitally complete*) iff every lower spring transversal sequence which in contained in $\mathcal{O}(x) := \{x, Tx, T^2(x), \dots\}$ for some $x \in X$ lower spring converges in X.

In connection with the preceding, the set $\mathcal{O}(x,\infty) := \{x, Tx, T^2x, \dots\}$ for $x \in X$ is called the **orbit** of x. A function f mapping X into reals is a f**-orbitally upper semicontinuous** at the point p iff for all sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to p$ $(n \to \infty)$ it follows that $f(p) \leq \liminf_{n\to\infty} f(x_n)$. A mapping $T : X \to X$ is said to be orbitally continuous if ξ, $x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x,\infty)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x,\infty))$.

Theorem 5. *Let T be mapping of a lower spring transversal space $X := (X, A)$ into itself and let X be lower spring T-orbitally complete. Suppose that there exists a function $\varphi : [0,+\infty] \to [0,+\infty]$ satisfying*

$$\text{(Id)} \qquad \Big(\forall t \in \mathbb{R}^0_+\Big)\Big(\varphi(t) > t \quad \textit{and} \quad \liminf_{z\to t-0} \varphi(z) > t\Big)$$

such that

$$\text{(J)} \qquad A(Tx, Ty) \geqslant \varphi\Big(\operatorname{trstdiam}\Big\{x, y, Tx, Ty, T^2x, T^2y, \dots\Big\}\Big)$$

for all $x, y \in X$. If $x \mapsto \operatorname{trstdiam}\mathcal{O}(x)$ or $x \mapsto A(x, Tx)$ is T-orbitally upper semicontinuous, then T has a unique fixed point $\zeta \in X$, and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to ζ for every $x \in X$.

A variant brief proof of this statement may be found in in the following sense with application of the following context from Lemma 6.18 by Tasković [28].

Annotation 1. We notice that in 1995 Tasković proved the following statement for a class of expansion mappings. Namely, if $X := (X, A)$ is a lower spring T-orbitally complete lower spring transversal space, if $T : X \to X$, and if there exists a number $q > 1$ such that

$$\text{(1)} \qquad A\Big(T(x), T(y)\Big) \geqslant qA(x, y)$$

for each $x, y \in X$, then T has a unique fixed point in the lower spring transversal space X.

Annotation 2. Let $X := (X, A_X)$ and $Y := (Y, A_Y)$ be two lower spring transversal spaces and let $T : X \to Y$. We notice, from: Tasković [28], that T be **lower spring continuous** at $x_0 \in X$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x \in X$ the following relation holds as

$$A_X(x, x_0) > \delta \quad \text{implies} \quad A_Y(T(x), T(x_0)) > \varepsilon.$$

A typical first example of a lower spring continuous mapping is the mapping $T : X \to X$ with property (1). For further facts on the lower spring continuous mappings see: Tasković [28].

Proof of Theorem 5. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ is a lower spring transversal Cauchy sequence. It is easy to verify that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ satisfies the following inequality

$$\operatorname{trstdiam}\mathcal{O}(T^{n+1}x) \geqslant \varphi(\operatorname{trstdiam}\mathcal{O}(T^n x))$$

for $n \in \mathbb{N}$, and hence applying Lemma 6.18 by Tasković [28] to the sequence $(\operatorname{trstdiam}\mathcal{O}(T^n x))$ we obtain that $\lim_{n\to\infty} \operatorname{trstdiam}\mathcal{O}(T^n x) = +\infty$. This implies that $\{T^n x\}_{n\in\mathbb{N}}$ is a lower spring transversal Cauchy sequence in X and, by lower spring T-orbital completeness, there is a $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \mapsto \operatorname{trstdiam} \mathcal{O}(x)$ is T-orbitally upper semicontinuous at ξ,

$$A(\xi, T\xi) \geqslant \operatorname{trstdiam} \mathcal{O}(\xi) \geqslant \liminf(\operatorname{trstdiam} \mathcal{O}(T^n x)) = +\infty;$$

thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ lower spring converges to a fixed point of T. On the other hand, if $x \mapsto A(x, Tx)$ is a T-orbitally lower semicontinuous at ξ we have

$$A(\xi, T\xi) \geqslant \liminf A(T^n x, T^{n+1} x) \geqslant \liminf(\operatorname{trstdiam} \mathcal{O}(T^n x)) = +\infty;$$

and thus again $T\xi = \xi$, i.e., we have again shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ lower spring converges to a fixed point of T.

We complete the proof by showing that T can have at most one fixed point: for, if $\xi \neq \eta$ were two fixed points, then

$$\begin{aligned} +\infty > \min\{A(\xi,\eta), A(\eta,\xi)\} &= \min\{A(T\xi,T\eta), A(T\eta,T\xi)\} \geqslant \\ &\geqslant \varphi\Big(\operatorname{trstdiam}\{\xi,\eta,T\xi,T\eta,T^2\xi,T^2\eta,\dots\}\Big) = \\ &= \varphi\Big(\min\{A(\xi,\xi), A(\eta,\eta), A(\xi,\eta), A(\eta,\xi)\}\Big) = \\ &= \varphi\Big(\min\{A(\xi,\eta), A(\eta,\xi)\}\Big) > \min\{A(\xi,\eta), A(\eta,\xi)\}, \end{aligned}$$

a contradiction. The proof is complete.

As immediate consequences of the preceding Theorem 5, we obtain directly the following interesting cases of (J):

(1) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\liminf_{z\to t-0} \psi(z) > t$ for every $t \in \mathbb{R}^0_+$ such that

$$A(Tx, Ty) \geqslant \psi(\operatorname{trstdiam}\{x, y, Tx, Ty\}) \ \text{ for all } \ x, y \in X.$$

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha > 1$ such that for all $x, y \in X$ the following inequality holds

$$A(Tx, Ty) \geqslant \alpha \operatorname{trstdiam}\{x, y, Tx, Ty\},$$

i.e., equivalently to

$$A(Tx, Ty) \geqslant \alpha \min\Big\{A(x,y), A(x,Tx), A(y,Ty), A(x,Ty), A(y,Tx)\Big\}.$$

(3) (*The condition of* $(m+k)$*-polygon*). There exists a constant $\alpha > 1$ such that for all $x, y \in X$ the following inequality holds in the form as

$$A(Tx, Ty) \geqslant \alpha \operatorname{trstdiam}\Big\{x, y, Tx, Ty, \dots, T^m x, T^k y\Big\}$$

for arbitrary fixed integers $m, k \geqslant 0$. (*This is a linear condition for trs.diameter of finite number of points*).

(4) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\liminf_{z\to t-0} \psi(z) > t$ for every $t \in \mathbb{R}^0_+$ such that

$$A(Tx, Ty) \geqslant \psi\Big(\operatorname{trstdiam}\{x, y, Tx, Ty, \dots, T^m x, T^k y\}\Big)$$

for arbitrary fixed integers $m, k \geqslant 0$ and for all $x, y \in X$. (*This is a nonlinear condition for trs.diameter of finite number of points*).

(5) There exists an increasing mapping $f : (\mathbb{R}_+^0)^5 \to \mathbb{R}_+^0$ satisfying the following condition in the form as $\lim_{z \to t-0} f(z,z,z,z,z) > t$ for every $t \in \mathbb{R}_+^0$ such that

$$A(Tx,Ty) \geqslant f\Big(A(x,y), A(x,Tx), A(y,Ty), A(x,Ty), A(y,Tx)\Big) \quad \text{for all} \quad x, y \in X.$$

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 5 in the following form.

Theorem 6. *Let T be a mapping of a lower spring transversal space $X := (X, A)$ into itself and let X be lower spring T-orbitally complete. Suppose that there exists a mapping $\varphi : [0, +\infty] \to [0, +\infty]$ satisfying (Id) such that*

$$\operatorname{trstdiam}\{Tx, T^2x, \ldots\} \geqslant \varphi\Big(\operatorname{trstdiam}\{x, Tx, T^2x, \ldots\}\Big)$$

for every $x \in X$. If $x \mapsto \operatorname{trstdiam} \mathcal{O}(x)$ or $x \mapsto A(x, Tx)$ is T-orbitally upper semicontinuous, then T has at least one fixed point in X.

The proof of this localization statement is totally analogous with the preceding proof of Theorem 5. Thus the proof of this result we omit.

Asymptotic contractions on lower spring transversal spaces. Let X be a nonempty set, $T : X \to X$, and let $A : X \times X \to \mathbb{R}_+^0 \cup \{+\infty\}$ be a given function. We shall introduce the concept of lower spring TCS-convergence in a space X, i.e., a lower spring transversal space $X := (X, A)$ satisfies the **condition of lower spring TCS-convergence** iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to +\infty$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n \in \mathbb{N}}$ has a convergent subsequence.

Theorem 7. *Let T be a mapping of lower spring transversal space $X := (X, A)$ into itself, where X satisfies the condition of lower spring TCS-convergence. Suppose that for all $x, y \in X$ there exist a sequence of nonnegative real functions $\{\alpha_n(x,y)\}_{n \in \mathbb{N}}$ such that $\alpha_n(x,y) \to +\infty$ $(n \to \infty)$ and positive integer $m(x,y)$ such that*

$$\text{(K)} \qquad A(T^n(x), T^n(y)) \geqslant \alpha_n(x,y) \quad \textit{for all} \quad n \geqslant m(x,y),$$

where $A : X \times X \to \mathbb{R}_+^0 \cup \{+\infty\}$. If $x \mapsto A(x, T(x))$ is a T-orbitally upper semicontinuous function, then T has a unique fixed point $\xi \in X$ and $T^n(x) \to \xi$ $(n \to \infty)$ for each $x \in X$.

Proof. For $y = T(x)$ from (K) we have that $A(T^n x, T^{n+1} x) \geqslant \alpha_n(x, Tx)$ for all $n \geqslant m(x, Tx)$), and thus we obtain that $A(T^n x, T^{n+1} x) \to +\infty$ $(n \to \infty)$. This implies (from lower spring TCS-convergence) that the sequence of iterates $\{T^n(x)\}_{n \in \mathbb{N}}$ has a convergent subsequence $\{T^{n(i)}(x)\}_{i \in \mathbb{N}}$ with the limit point $\xi \in X$. Since $x \mapsto A(x, T(x))$ is T orbitally upper semicontinuous, we get

$$A(\xi, T(\xi)) \geqslant \limsup_{i\to\infty} A(T^{n(i)}x, T^{n(i)+1}x) = \limsup_{n\to\infty} A(T^n x, T^{n+1}x) = +\infty,$$

which implies that $A(\xi, T(\xi)) = +\infty$, i.e., $\xi = T(\xi)$. We complete the proof by showing that T can have at most one fixed point. Indeed, if we suppose that $\xi \neq \eta$ were two fixed points, then from (K) we have

$$A(\xi, \eta) = A(T^n(\xi), T^n(\eta)) \geqslant \alpha_n(\xi, \eta) \text{ for every } n \geqslant m(\xi, \eta);$$

taking limits as $n \to \infty$ we obtain a contradiction. Thus we obtain that $\xi = \eta$, i.e., T has a unique fixed point in X. The proof is complete.

Note that, from the preceding proof of Theorem 7, we can give the following local form of this statement.

Theorem 8. (Localization of (K)). *Let T be a mapping of lower spring transversal space $X := (X, A)$ into itself, where X satisfies the condition of lower spring TCS-convergence. Suppose that for each $x \in X$ there exist a sequence of nonnegative real functions $\{\alpha_n(x, Tx)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, Tx) \to +\infty$ $(n \to \infty)$ and positive integer $m(x)$ such that*

$$A(T^n(x), T^{n+1}(x)) \geqslant \alpha_n(x, Tx) \quad \textit{for all} \quad n \geqslant m(x),$$

where $A : X \times X \to \mathbb{R}_+^0 \cup \{+\infty\}$. If $x \mapsto A(x, T(x))$ is a T-orbitally upper semicontinuous function, then T has at least one fixed point in X.

The proof of this statement is totally analogous with the preceding proof of Theorem 7. Thus the proof of this result we omit.

Middle transversal spring spaces. In the preceding part of this paper I have had two spaces (or two sides of a space): the upper transversal spring space and the lower transversal spring space. As a new space (or as third side of a given space) is a middle transversal spring space by Tasković [28]. In this sense, a **middle transversal spring space** is an upper transversal spring space and a lower transversal spring space simultaneous.

Annotation. >From the facts by Tasković [28] we have a main result of the form as: that every space, de facto, has three sides; which in this case I denoted with as: the upper transversal spring space, the lower transversal spring space, and the middle transversal spring space!

This in further considerations of the middle transversal spring spaces we esteem all the preceding facts on upper and lower transversal spring space!

The equation $x = T(x, \dots, x)$. In 1980 I have been proved a result of fixed point on metric spaces which has a best long of all known sufficiently conditions for the existing of unique fixed point, cf. Tasković [15]. This result of Theorem 1 is a generalization a great number of known results.

This statement is well-known as "a finest theorem of nonlinear functional analysis" for metric spaces. In this part of this paper we give Theorem 1 on transversal upper and lower spring spaces and its applications also.

Annotation 3. We notice that in 1975 Tasković proved the following statement for a class of contraction mappings. Namely, if $X := (X, A)$ is an upper spring T-orbitally complete upper spring transversal space, if $T : X \to X$, and if there exists a number $0 \leqslant q < 1$ such that

$$A\Big(T(x), T(y)\Big) \leqslant qA(x, y) \tag{2}$$

for each $x, y \in X$, then T has a unique fixed point in the upper spring transversal space X.

Annotation 4. Let $X := (X, A_X)$ and $Y := (Y, A_Y)$ be two upper spring transversal spaces and let $T : X \to Y$. We notice, from: Tasković [28], that T be **upper spring continuous** at $x_0 \in X$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x \in X$ the following relation holds as

$$A_X(x, x_0) < \delta \quad \text{implies} \quad A_Y(T(x), T(x_0)) < \varepsilon.$$

A typical first example of an upper spring continuous mapping is the mapping $T : X \to X$ with property (2). For further facts on the upper spring continuous mappings see: Tasković [28].

In this part of the paper we consider a form of Theorem 1 on cartesian product of upper spring transversal spaces in the following context.

Let X be an arbitrary upper spring transversal space. By Tasković [16] and [33] for a mapping $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number) we will construct the *iteration sequence* $\{T^n(u)\}_{n\in\mathbb{N}}$ for an arbitrary point $u = (u_1, \ldots, u_k) \in X^k$ in the following sense. Let T^0 =Identical mapping and

$$T^n := T\psi^{n-1} \quad (n = 1, 2, \ldots), \tag{Is}$$

where $\psi : X^k \to X^k$ defined by $\psi(u_1, \ldots, u_k) = (u_2, \ldots, u_{k+1})$ for $u_{k+1} = T(u_1, \ldots, u_k)$ and ψ^0 =Identical mapping.

We are now in a position to formulate the following statement for mappings of cartesian product upper spring transversal spaces.

Let, in further, $\mathcal{O}(x, T(x)) := \{x_k, T(x), T^2(x), \ldots\}$ for $x = (x_1, \ldots, x_k)$ and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Also, $\mathcal{O}(t, T(t, \ldots, t)) := \{t, T(t, \ldots, t), T^2(t, \ldots, t), \ldots\}$. A function $t \mapsto A(t, T(t, \ldots, t))$ is **T-orbitally lower semicontinuous** at $p \in X$ iff $T^n(x) \to p$ ($n \to \infty$) implies that $A(p, T(p, \ldots, p)) \leqslant \liminf_{n\to\infty} A(T^n(x), T(T^n(x), \ldots, T^{n+k-1}(x)))$.

Theorem 9. *Let $X := (X, A)$ be an upper spring orbitally complete upper spring transversal space and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying*

$$\Big(\forall t \in \mathbb{R}_+\Big)\left(\varphi(t) < t \text{ and } \limsup_{z\to t+0} \varphi(z) < t\right) \tag{Iφ}$$

such that

$$A(Tx, Ty) \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_k, y_k, Tx, Ty, T^2x, T^2y, \ldots\Big\}\Big) \tag{G}$$

and $\operatorname{trstdiam} \mathcal{O}(x, Tx) \in \mathbb{R}^0_+$ *for all* $x = (x_1, \dots, x_k)$ *and* $y = (y_1, \dots, y_k)$ *in* X^k. *If in the following context* $t \mapsto \operatorname{trstdiam} \mathcal{O}(t, T(t, \dots, t))$ *or* $t \mapsto A(t, T(t, \dots, t))$ *is* T*-orbitally lower semicontinuous, then the following equation of the form*

$$\text{(Eq)} \qquad u = T(u, \dots, u), \quad u \in X,$$

has a unique solution $\xi \in X$ *and the sequence of iterates* $\{T^n(x)\}_{n \in \mathbb{N}}$ *converges to* ξ *for every* $x = (x_1, \dots, x_k) \in X^k$, *where*

$$T^n(x) := x_{n+k} = T(x_n, \dots, x_{n+k-1}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Proof. Let $x = (x_1, \dots, x_k)$ be an arbitrary point in X^k. We can show that the sequence of iterates $\{T^n(x)\}_{n \in \mathbb{N}}$ is a transversal upper spring Cauchy sequence. For each $n \in \mathbb{N}$, let

$$\text{(3)} \qquad \delta_n := \sup \Big\{ A\Big(T^i x, T^j x\Big) : i, j \geqslant n \Big\},$$

then, by the facts of this statement, $\delta_n < +\infty$. Since δ_n $(n \in \mathbb{N})$ is a nonincreasing sequence in $\mathbb{R}^0_+$, there is an $\delta \geqslant 0$ such that $\delta_n \to \delta$ $(n \to \infty)$. We claim that $\delta = 0$. It is easy to verify that the sequence $\{T^n(x)\}_{n \in \mathbb{N}}$ satisfies, from (G) for $\delta_n := \operatorname{trstdiam} \mathcal{O}(T^n(x))$, the following inequality in the form as

$$\operatorname{trstdiam} \mathcal{O}(T^{n+1}(x)) \leqslant \varphi\Big(\operatorname{trstdiam} \mathcal{O}(T^n(x))\Big)$$

for $n \in \mathbb{N}$, and hence applying Lemma 1 by Tasković [32] to the sequence $(\operatorname{trstdiam} \mathcal{O}(T^n(x)))$ we obtain that $\delta = \lim_{n \to \infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = 0$. This implies $\{T^n(x)\}_{n \in \mathbb{N}}$ is a transversal upper spring Cauchy sequence in X and, by upper spring T-orbitally completeness, there is a $\xi \in X$ such that $T^n(x) \to \xi$ $(n \to \infty$, $x = (x_1, \dots, x_k))$, where $T^n(x) := x_{n+k} = T(x_n, \dots, x_{n+k-1})$.

Let us prove that ξ satisfied the equation of the form (Eq), in the sense precised above. First, since $t \mapsto \operatorname{trstdiam} \mathcal{O}(t, T(t, \dots, t))$ is T-orbitally lower semicontinuous at ξ, we obtain

$$A(\xi, T(\xi, \dots, \xi)) \leqslant \operatorname{trstdiam} \mathcal{O}(\xi, T(\xi, \dots, \xi)) \leqslant$$
$$\leqslant \liminf_{n \to \infty} \operatorname{trstdiam} \mathcal{O}\Big(T^n(x), T(T^n(x), \dots, T^{n+k-1}(x))\Big) =$$
$$= \liminf_{n \to \infty} \operatorname{trstdiam} \mathcal{O}\Big(T^n(x), T^{n+k}(x)\Big) \leqslant \liminf_{n \to \infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = 0;$$

thus $\xi = T(\xi, \dots, \xi)$, and we have shown that in this case for each $x \in X^k$ the sequence $\{T^n(x)\}_{n \in \mathbb{N}}$ converges to a solution of equation (Eq).

On the other hand, if $t \mapsto A(t, T(t, \ldots, t))$ is T-orbitally lower semicontinuous at ξ, we have the following

$$A(\xi, T(\xi, \ldots, \xi)) \leqslant \liminf_{n\to\infty} A\Big(T^n(x), T(T^{n+1}(x), \ldots, T^{n+k}(x))\Big) =$$
$$= \liminf_{n\to\infty} A\Big(T^n(x), T(\psi(T^n(x), \ldots, T^{n+k-1}(x)))\Big) =$$
$$= \liminf_{n\to\infty} A(T^n(x), T^{n+k}(x)) \leqslant \liminf_{n\to\infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = 0;$$

and thus again $\xi = T(\xi, \ldots, \xi)$, i.e., we have again shown that for each $x \in X^k$ the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ upper converges to a solution of equation (Eq).

We complete the proof by showing that equation (Eq) can have at most one solution: for, if $\xi = T(\xi, \ldots, \xi) \neq \eta = T(\eta, \ldots, \eta)$ were two solutions of (Eq), then

$$0 < \max\Big\{A(\xi, \eta), A(\eta, \xi)\Big\} =$$
$$= \max\Big\{A\Big(T(\xi, \ldots, \xi), T(\eta, \ldots, \eta)\Big), A\Big(T(\eta, \ldots, \eta), T(\xi, \ldots, \xi)\Big)\Big\} \leqslant$$
$$\leqslant \varphi\Big(\operatorname{trstdiam}\Big\{\xi, \eta, T(\xi, \ldots, \xi), T(\eta, \ldots, \eta), T^2(\xi, \ldots, \xi), T^2(\eta, \ldots, \eta), \ldots\Big\}\Big) =$$
$$= \varphi\Big(\operatorname{trstdiam}\Big\{\xi, \eta, \xi, \ eta, \ldots\Big\}\Big) = \varphi\Big(\max\Big\{A(\xi, \eta), A(\xi, \xi), A(\eta, \eta), A(\eta, \xi)\Big\}\Big) <$$
$$< \max\Big\{A(\xi, \eta), A(\eta, \xi)\Big\},$$

a contradiction, i.e., $\xi = \eta = T(\xi, \ldots, \xi)$ is a unique solution of equation (Eq). The proof is complete.

Theorem 10. *Let $X := (X, A)$ be an upper spring orbitally complete upper spring transversal space and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying* (Iφ) *such that* (G) *or*

$$\text{(G')} \qquad A(T^2(x), T^2(y)) \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{T(x), T(y), T^2(x), T^2(y), \ldots\Big\}\Big)$$

and $\operatorname{trstdiam} \mathcal{O}(T(x), T^2(x), \ldots) \in \mathbb{R}^0_+$ *for all $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in X^k. If $(r, t) \to A(r, t)$ is continuous, then the equation* (Eq) *has a unique solution $\xi \in X$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to ξ for each $x = (x_1, \ldots, x_k) \in X^k$, where*

$$T^n(x) := x_{n+k} = T(x_n, \ldots, x_{n+k-1}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Proof. The proof of this statement for convergence of the sequence of the form as in the following ($\operatorname{trstdiam} \mathcal{O}\ (T^n(x))$) is a totally analogous with the preceding proof of Theorem 9, and a new application of Lemma 1 by Tasković [32]. Thus, as in the case of Theorem 9, we obtain that the sequence δ_n ($n \in \mathbb{N}$) from (3) is a nonincreasing sequence in $\mathbb{R}^0_+$, and thus there is an $\delta \geqslant 0$ such that $\delta_n \to \delta$ ($n \to \infty$). We claim that $\delta = 0$. It is easy to verify that the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ satisfies, from (G) or (G') for $\delta_n := \operatorname{trstdiam} \mathcal{O}(T^n(x))$, in the form as

$$\operatorname{trstdiam} \mathcal{O}(T^{n+1}(x)) \leqslant \varphi\Big(\operatorname{trstdiam} \mathcal{O}(T^n(x))\Big)$$

for $n \in \mathbb{N}$, and hence applying Lemma 1 by Tasković [32] to the sequence $(\operatorname{trstdiam} \mathcal{O}(T^n(x)))$ we obtain again $\delta = \lim_{n\to\infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = 0$. This implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ is a transversal upper spring Cauchy sequence in X and, by the upper spring T-orbitally completeness, there is a $\xi \in X$ such that $T^n(x) \to \xi$ ($n \to \infty$, $x = (x_1, \dots, x_k)$), where $T^n(x) := x_{n+k} = T(x_n, \dots, x_{n+k-1})$.

Let us prove that ξ satisfied the equation of the form (Eq), in the sense precised above. We get, according to our hypothesis on T,

$$\begin{aligned} \max\Big\{A\Big(x_{n+k+1}, T(\xi,\dots,\xi)\Big), A\Big(T(\xi,\dots,\xi), \xi_{n+k+1}\Big)\Big\} &= \\ = \max\Big\{A\Big(T(x_{n+1},\dots,x_{n+k}), T(\xi,\dots,\xi)\Big), &\\ A\Big(T(\xi,\dots,\xi), T(x_{n+1},\dots,x_{n+k})\Big)\Big\} &\leqslant \\ \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1}, \dots, T(\xi,\dots,\xi), T^2(\xi,\dots,\xi), \dots\Big\}\Big), \end{aligned}$$

or

$$\begin{aligned} \max\Big\{A\Big(x_{n+k+1}, T(\xi,\dots,\xi)\Big), A\Big(T(\xi,\dots,\xi), x_{n+k+1}\Big)\Big\} &= \\ = \max\Big\{A\Big(T^2(x_n,\dots,x_{n+k-1}), T^2(\xi,\dots,\xi)\Big), &\\ A\Big(T^2(\xi,\dots,\xi), T^2(x_n,\dots,x_{n+k-1})\Big)\Big\} &\leqslant \\ \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1}, \dots, T(\xi,\dots,\xi), T^2(\xi,\dots,\xi), \dots\Big\}\Big), \end{aligned}$$

and thus, by the facts of statement, also we obtain the following inequalities of the form as

$$\begin{aligned} 0 < t = \max\Big\{A(\xi, T(\xi,\dots,\xi)), A(T(\xi,\dots,\xi), \xi)\Big\} &= \\ = \lim_{n\to\infty} \max\Big\{A\Big(x_{n+k}, T(\xi,\dots,\xi)\Big), A\Big(T(\xi,\dots,\xi), x_{n+k}\Big)\Big\} &\leqslant \\ \leqslant \limsup_{n\to\infty} \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1}, \dots, T(\xi,\dots,\xi), \dots\Big\}\Big) &\leqslant \\ \leqslant \limsup_{z\to t+0} \varphi(z) < t = \max\Big\{A(\xi, T(\xi,\dots,\xi)), A(T(\xi,\dots,\xi), \xi)\Big\} & \end{aligned}$$

which is a contradiction, i.e., which was to be proved. Uniqueness follows immediately from the following inequalities, i.e., if $T(\xi, \dots, \xi) = \xi \neq \eta =$

$T(\eta, \ldots, \eta)$, then

$$0 < \max\Big\{A(\xi,\eta), A(\eta,\xi)\Big\} =$$

$$= \max\Big\{A\Big(T(\xi,\ldots,\xi), T(\eta,\ldots,\eta)\Big), A\Big(T(\eta,\ldots,\eta), T(\xi,\ldots,\xi)\Big)\Big\} \leqslant$$

$$\leqslant \varphi\Big(\operatorname{trstdiam}\Big\{\xi,\eta,T(\xi,\ldots,\xi),T(\eta,\ldots,\eta),$$

$$T^2(\xi,\ldots,\xi), T^2(\eta,\ldots,\eta),\ldots\Big\}\Big) =$$

$$= \varphi\Big(\max\Big\{A(\xi,\eta), A(\xi,\xi), A(\eta,\xi), A(\eta,\eta)\Big\}\Big) < \max\Big\{A(\xi,\eta), A(\eta,\xi)\Big\},$$

or

$$0 < \max\Big\{A(\xi,\eta), A(\eta,\xi)\Big\} =$$

$$= \max\Big\{A\Big(T^2(\xi,\ldots,\xi), T^2(\eta,\ldots,\eta)\Big), A\Big(T^2(\eta,\ldots,\eta), T^2(\xi,\ldots,\xi)\Big)\Big\} \leqslant$$

$$\leqslant \varphi\Big(\operatorname{trstdiam}\Big\{T(\xi,\ldots,\xi),T(\eta,\ldots,\eta),T^2(\xi,\ldots,\xi), T^2(\eta,\ldots,\eta),\ldots\Big\}\Big) =$$

$$= \varphi\Big(\max\Big\{A(\xi,\eta), A(\xi,\xi), A(\eta,\xi), A(\eta,\eta)\Big\}\Big) < \max\Big\{A(\xi,\eta), A(\eta,\xi)\Big\},$$

a contradiction, i.e., $\eta = \xi = T(\xi, \ldots, \xi)$ is a unique solution of the equation (Eq). The proof is complete.

As immediate consequences of the preceding Theorem 9, we obtain directly the following interesting cases of (G):

(1) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z\to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant \psi\Big(\operatorname{trstdiam}\Big\{x_k, y_k, Tx, Ty\Big\}\Big)$$

for all $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in X^k.

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha \in [0, 1)$ such that for all $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in X^k the following inequality holds

$$A(Tx, Ty) \leqslant \alpha \operatorname{trstdiam}\{x_k, y_k, Tx, Ty\},$$

i.e., equivalently to

$$A(Tx, Ty) \leqslant \alpha \max\Big\{A(x_k, y_k), A(x_k, Tx), A(y_k, Ty), A(x_k, Ty), A(y_k, Tx)\Big\}.$$

(3) (*The condition of* $(m + r)$*-polygon*). There exists a constant $\alpha \in [0, 1)$ such that for all $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in X^k the following inequality holds in the form as

$$A(Tx, Ty) \leqslant \alpha \operatorname{trstdiam}\Big\{x_k, y_k, Tx, Ty, \ldots, T^m x, T^r y\Big\}$$

for arbitrary fixed integers $m, r \geqslant 0$. (*This is a linear condition for trs.diameter of finite number of points*).

(4) There exists a nondecreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying the following condition in the form as $\limsup_{z \to t+0} \psi(z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant \psi\Big(\operatorname{trstdiam}\{x_k, y_k, Tx, Ty, \dots, T^m x, T^r y\}\Big)$$

for arbitrary fixed integers $m, r \geqslant 0$ and for all $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in X^k. (*This is a nonlinear condition for trs.diameter of finite number of points*).

(5) There exists an increasing mapping for any coordinates of $f : (\mathbb{R}^0_+)^5 \to \mathbb{R}^0_+$ satisfying the following inequality $\limsup_{z \to t+0} f(z, z, z, z, z) < t$ for every $t \in \mathbb{R}_+$ such that

$$A(Tx, Ty) \leqslant f\Big(A(x_k, y_k), A(x_k, Tx), A(y_k, Ty), A(x_k, Ty), A(y_k, Tx)\Big)$$

for all $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in X^k.

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 9 in the following form.

Theorem 11. *Let $X := (X, A)$ be an upper spring orbitally complete upper spring transversal space and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying* (Iφ) *such that*

$$\operatorname{trstdiam}\Big\{Tx, T^2x, \dots\Big\} \leqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_k, Tx, T^2x, \dots\Big\}\Big)$$

and $\operatorname{trstdiam}\mathcal{O}(x, Tx) \in \mathbb{R}^0_+$ *for every $x = (x_1, \dots, x_k)$ in X^k. If $t \mapsto$* $\operatorname{trstdiam}\mathcal{O}(t, T(t, \dots, t))$ *or $t \mapsto A(t, T(t, \dots, t))$ is T-orbitally lower semicontinuous, then there exists at least one solution of the equation* (Eq).

The proof of this localization statement is totally analogous with the preceding proof of Theorem 9. Thus the proof of this result we omit.

As two immediate consequence of the preceding statements we have former results in the following bookings.

Corollary 1. (Localization result, Tasković [34]). *Let (X, ρ) be a complete metric space and let T be a mapping of X^k ($k \in \mathbb{N}$ is a fixed number) into X satisfying the condition*

$$\rho\Big[T(u_1, \dots, u_k), T(u_2, \dots, u_{k+1})\Big] \leqslant f\Big(\alpha_1 \rho[u_1, u_2], \dots, \alpha_k \rho[u_k, u_{k+1}]\Big)$$

for all $u_1, \dots, u_k, u_{k+1} \in X$, where $f : (\mathbb{R}^0_+)^k \to \mathbb{R}^0_+$ is an increasing, semihomogeneous mapping such that $f(\alpha_1, \dots, \alpha_k) \in [0, 1)$ and $x \mapsto f(\alpha_1 x, \dots, \alpha_k x^k)$ is continuous at the point $x = 1$ and α_i ($i = 1, \dots, k$) are nonnegative real constants. Then:

(a) *There exists a point $\xi \in X$ as a solution of the equation $x = T(x, \dots, x)$ and it is unique when $f(\alpha_1, 0, \dots, 0) + \dots + f(0, \dots, 0, \alpha_k) < 1$.*

(b) *The point $\xi \in X$ is the limit of the following sequence $\{x_n\}_{n \in \mathbb{N}}$ which is defined in the following sense as*

$$x_{n+k} = T(x_n, \dots, x_{n+k-1}), \quad n \in \mathbb{N}, \tag{4}$$

independently of initial values $x_1, \dots, x_k \in X$.

(c) *The rapidity of convergence of the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by* (66) *to the point* $\xi \in X$ *is evaluated for* $\theta \in (0,1)$ *by the following inequality in the form as*:

$$\rho[x_{n+k},\xi] \leqslant \theta^n(1-\theta)^{-1}\max_{i=1,\dots,k}\Big\{\rho[x_i,x_{i+1}]\theta^{-i}\Big\},\quad n\in\mathbb{N}.$$

Corollary 2. (Tasković [15]). *Let* (X,ρ) *be a complete metric space and let* T *be a mapping of* X^k ($k\in\mathbb{N}$ *is a fixed number*) *into* X *such that for all* $x,y\in X^k$ *there exist nonnegative numbers* $\alpha(x,y)$, $\beta(x,y)$, $\gamma(x,y)$, $\delta(x,y)$ *and* $q_i(x,y)$ *for* $i=1,\dots,k$ *satisfying*

$$\sup_{x,y\in X^k}\left\{\alpha(x,y)+\beta(x,y)+2\gamma(x,y)+\delta(x,y)+\sum_{i=1}^k q_i(x,y)\right\}\in[0,1)$$

and

$$\rho[Tx,Ty]\leqslant \alpha(x,y)\rho[u_k,Tx]+\beta(x,y)\rho[v_k,Ty]+$$
$$+\gamma(x,y)\rho[u_k,Ty]+\delta(x,y)\rho[v_k,Tx]+\sum_{i=1}^k q_i(x,y)\rho[u_i,u_{i+1}]$$

for all $x=(u_1,\dots,u_k)$ *and* $y=(v_1,\dots,v_k)$ *in* X^k. *Then the following facts hold in the form of the following bookings*:

(a) *The equation* $x=T(x,\dots,x)$ *has a unique solution* $\xi\in X$ *and the point* ξ *is the limit of the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$x_{n+k}=T(x_n,\dots,x_{n+k-1}),\quad n\in\mathbb{N}, \tag{5}$$

independently of initial values $x_1,\dots,x_k\in X$.

(b) *The rapidity of convergence of the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by (5) to the point* $\xi\in X$ *is evaluated for* $\theta\in(0,1)$ *by*

$$\rho[x_{n+k},\xi] \leqslant \theta^n(1-\theta)^{-1}\max_{i=1,\dots,k}\Big\{\rho[x_i,x_{i+1}]\theta^{-i}\Big\},\quad n\in\mathbb{N}.$$

The lower spring transversal spaces. In this part of the paper we give the following results of fixed point on cartesian product of lower spring transversal spaces.

Theorem 12. *Let* $X:=(X,A)$ *be a lower spring orbitally complete lower spring transversal space and* $T:X^k\to X$ ($k\in\mathbb{N}$ *is a fixed number*). *Suppose that there exists a function* $\varphi:[0,+\infty]\to[0,+\infty]$ *satisfying*

$$\Big(\forall t\in\mathbb{R}^0_+\Big)\Big(\varphi(t)>t\quad\text{and}\quad\liminf_{z\to t-0}\varphi(z)>t\Big) \tag{Id}$$

such that

$$A(Tx,Ty)\geqslant\varphi\Big(\operatorname{trstdiam}\Big\{x_k,y_k,Tx,Ty,T^2x,T^2y,\dots\Big\}\Big) \tag{E}$$

for all $x = (x_1, \ldots, x_k)$ *and* $y = (y_1, \ldots, y_k)$ *in* X^k. *If* $t \mapsto \operatorname{trstdiam} \mathcal{O}(t, T(t, \ldots, t))$ *or* $t \mapsto A(t, T(t, \ldots, t))$ *is* T*-orbitally upper semicontinuous, then the following equation of the form*

$$\text{(Ea)} \qquad u = T(u, \ldots, u), \quad u \in X,$$

has a unique solution $\xi \in X$ *and the sequence of iterates* $\{T^n(x)\}_{n\in\mathbb{N}}$ *converges to* ξ *for every* $x = (x_1, \ldots, x_k) \in X^k$, *where*

$$T^n(x) := x_{n+k} = T(x_n, \ldots, x_{n+k-1}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Proof. Let $x = (x_1, \ldots, x_k)$ be an arbitrary point in X^k. We can show then that the sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ is a transversal lower spring Cauchy sequence. It is easy to verify that the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ satisfies the following inequality

$$\operatorname{trstdiam} \mathcal{O}(T^{n+1}(x)) \geqslant \varphi\Big(\operatorname{trstdiam} \mathcal{O}(T^n(x))\Big)$$

for $n \in \mathbb{N}$, and hence applying Lemma 6.18 by Tasković [28] to the sequence $(\operatorname{trstdiam} \mathcal{O}(T^n(x)))$ we obtain $\lim_{n\to\infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = +\infty$. This implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ is a lower transversal spring Cauchy sequence in X and, by lower spring T-orbitally completeness, there is a $\xi \in X$ such that $T^n(x) \to \xi$ $(n \to \infty, x = (x_1, \ldots, x_k))$. Since $t \mapsto \operatorname{trstdiam} \mathcal{O}(t, T(t, \ldots, t))$ is T-orbitally upper semicontinuous at ξ, we obtain

$$A(\xi, T(\xi, \ldots, \xi)) \geqslant \operatorname{trstdiam} \mathcal{O}(\xi, T(\xi, \ldots, \xi)) \geqslant$$
$$\geqslant \limsup_{n\to\infty} \operatorname{trstdiam} \mathcal{O}\Big(T^n(x), T\big(T^n(x), \ldots, T^{n+k-1}(x)\big)\Big) =$$
$$= \limsup_{n\to\infty} \operatorname{trstdiam} \mathcal{O}\big(T^n(x), T^{n+k}(x)\big) \geqslant \limsup_{n\to\infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = +\infty;$$

thus $\xi = T(\xi, \ldots, \xi)$, and we have shown that in this case for each $x \in X^k$ the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to a solution of equation (Ea). If, in the case that, $t \mapsto A(t, T(t, \ldots, t))$ is T-orbitally upper semicontinuous at ξ, we obtain

$$A(\xi, T(\xi, \ldots, \xi)) \geqslant \limsup_{n\to\infty} A\Big(T^n(x), T\big(T^n(x), \ldots, T^{n+k-1}(x)\big)\Big) =$$
$$= \limsup_{n\to\infty} A(T^n(x), T^{n+k}(x)) \geqslant \limsup_{n\to\infty} \operatorname{trstdiam} \mathcal{O}(T^n(x)) = +\infty;$$

and thus again $\xi = T(\xi, \ldots, \xi)$, i.e., we have in this case that for each $x \in X^k$ the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ lower converges to a solution of equation (Ea).

We complete the proof by showing that the equation (Ea) can have at most one solution: for, if $\xi = T(\xi, \ldots, \xi) \neq \eta = T(\eta, \ldots, \eta)$ were two solution of

(Ea), then

$$+\infty > \min\Big\{A(\xi,\eta), A(\eta,\xi)\Big\} =$$

$$= \min\Big\{A\Big(T(\xi,\ldots,\xi), T(\eta,\ldots,\eta)\Big), A\Big(T(\eta,\ldots,\eta), T(\xi,\ldots,\xi)\Big)\Big\} \geqslant$$

$$\geqslant \varphi\Big(\operatorname{trstdiam}\Big\{\xi,\eta,T(\xi,\ldots,\xi),T(\eta,\ldots,\eta),T^2(\xi,\ldots,\xi),T^2(\eta,\ldots,\eta),\ldots\Big\} =$$

$$= \varphi\Big(\operatorname{trstdiam}\{\xi,\eta,\xi,\eta,\ldots\}\Big) =$$

$$= \varphi\Big(\min\{A(\xi,\eta),A(\xi,\xi),A(\eta,\xi),A(\eta,\eta)\}\Big) > \min\big\{A(\xi,\eta),A(\eta,\xi)\big\},$$

a contradiction, i.e., $\xi = \eta = T(\xi,\ldots,\xi)$ is a unique solution of equation (Ea). The proof is complete.

Theorem 13. *Let $X := (X, A)$ be a lower spring orbitally complete lower spring transversal space and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Suppose that there exists a function $\varphi : [0,+\infty] \to [0,+\infty]$ satisfying* (Id) *such that* (E) *or*

$$\text{(E')}\qquad A\Big(T^2(x), T^2(y)\Big) \geqslant \varphi\Big(\operatorname{trstdiam}\Big\{T(x),T(y),T^2(x),T^2(y),\ldots\Big\}\Big)$$

for all $x = (x_1,\ldots,x_k)$ and $y = (y_1,\ldots,y_k)$ in X^k. If $(r,t) \mapsto A(r,t)$ is continuous, then the equation (Ea) *has a unique solution $\xi \in X$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to ξ for each $x = (x_1,\ldots,x_k) \in X^k$, where*

$$T^n(x) := x_{n+k} = T(x_n,\ldots,x_{n+k-1}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Proof. Let $x = (x_1,\ldots,x_k) \in X^k$ be an arbitrary point. As in the preceding proofs, with the totally analogous, we show, application of Lemma 6.18 by Tasković [28], that the sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ is a transversal lower spring Cauchy sequence. Thus, by lower spring T-orbitally completeness, there is $\xi \in X$ such that $T^n(x) \to \xi$ ($n \to \infty$, $x = (x_1,\ldots,x_k)$). We get, according to our hypothesis on T,

$$\min\Big\{A(x_{n+k+1}, T(x,\ldots,\xi)), A(T(\xi,\ldots,\xi), x_{n+k+1})\Big\} =$$

$$= \min\Big\{A(T(x_{n+1},\ldots,x_{n+k}), T(\xi,\ldots,\xi)),$$

$$A(T(\xi,\ldots,\xi), T(x_{n+1},\ldots,x_{n+k}))\Big\} \geqslant$$

$$\geqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1},\ldots,T(\xi,\ldots,\xi),T^2(\xi,\ldots,\xi),\ldots\Big\}\Big),$$

or

$$\min\Big\{A(x_{n+k+1}, T(\xi, \dots, \xi)), A(T(\xi, \dots, \xi), x_{n+k+1})\Big\} =$$

$$= \min\Big\{A(T^2(x_n, \dots, x_{n+k-1}), T^2(\xi, \dots, \xi)),$$

$$A(T^2(\xi, \dots, \xi), T^2(x_n, \dots, x_{n+k-1}))\Big\} \geqslant$$

$$\geqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1}, \dots, T(\xi, \dots, \xi), T^2(\xi, \dots, \xi), \dots\Big\}\Big),$$

and thus, by the facts of statement, also we obtain the following inequalities of the form as

$$+\infty > t = \min\Big\{A(\xi, T(\xi, \dots, \xi)), A(T(\xi, \dots, \xi), \xi)\Big\} =$$

$$= \lim_{n\to\infty} \min\Big\{A(x_{n+k}, T(\xi, \dots, \xi)), A(T(\xi, \dots, \xi), x_{n+k})\Big\} \geqslant$$

$$\geqslant \liminf_{n\to\infty} \varphi\Big(\operatorname{trstdiam}\Big\{x_{n+k}, x_{n+k+1}, \dots, T(\xi, \dots, \xi), \dots\Big\}\Big) \geqslant$$

$$\geqslant \liminf_{z\to t-0} \varphi(z) > t = \min\Big\{A(\xi, T(\xi, \dots, \xi)), A(T(\xi, \dots, \xi), \xi)\Big\},$$

which is a contradiction, i.e., which was to be proved. Uniqueness follows immediately from the following inequalities, i.e., if $T(\xi, \dots, \xi) = \xi \neq \eta = T(\eta, \dots, \eta)$, then

$$+\infty > \min\Big\{A(\xi, \eta), A(\eta, \xi)\Big\} =$$

$$= \min\Big\{A(T(\xi, \dots, \xi), T(\eta, \dots, \eta)), A(T(\eta, \dots, \eta), T(\xi, \dots, \xi))\Big\} \geqslant$$

$$\geqslant \varphi\Big(\operatorname{trstdiam}\Big\{\xi, \eta, T(\xi, \dots, \xi), T(\eta, \dots, \eta), T^2(\xi, \dots, \xi), T^2(\eta, \dots, \eta), \dots\Big\}\Big) =$$

$$= \varphi\Big(\min\Big\{A(\xi, \eta), A(\xi, \xi), A(\eta, \xi), A(\eta, \eta)\Big\}\Big) > \min\Big\{A(\xi, \eta), A(\eta, \xi)\Big\},$$

or

$$+\infty > \min\Big\{A(\xi, \eta), A(\eta, \xi)\Big\} =$$

$$= \min\Big\{A(T^2(\xi, \dots, \xi), T^2(\eta, \dots, \eta)), A(T^2(\eta, \dots, \eta), T^2(\xi, \dots, \xi))\Big\} \geqslant$$

$$\geqslant \varphi\Big(\operatorname{trstdiam}\Big\{T(\xi, \dots, \xi), T(\eta, \dots, \eta), T^2(\xi, \dots, \xi), T^2(\eta, \dots, \eta), \dots\Big\}\Big) =$$

$$= \varphi\Big(\min\Big\{A(\xi, \eta), A(\xi, \xi), A(\eta, \xi), A(\eta, \eta)\Big\}\Big) > \min\Big\{A(\xi, \eta), A(\eta, \xi)\Big\},$$

a contradiction, i.e., $\xi = \eta = T(\xi, \dots, \xi)$ is a unique solution of the equation (Ea). The proof is complete.

As immediate consequences of the preceding Theorem 12, we obtain directly the following interesting cases of (E):

(1) There exists a nondecreasing function $\psi : [0,+\infty] \to [0,+\infty]$ satisfying the following condition in the form as $\liminf_{z\to t-0} \psi(z) > t$ for every $t \in \mathbb{R}^0_+$ such that

$$A(Tx,Ty) \geqslant \psi\Big(\operatorname{trstdiam}\Big\{x_k, y_k, Tx, Ty\Big\}\Big)$$

for all $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ in X^k.

(2) (*Special case of* (1) *for* $\psi(t) = \alpha t$). There exists a constant $\alpha > 1$ such that for all $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ in X^k the following inequality holds

$$A(Tx,Ty) \geqslant \alpha \operatorname{trstdiam}\{x_k, y_k, Tx, Ty\},$$

i.e., equivalently to

$$A(Tx,Ty) \geqslant \alpha \min\Big\{A(x_k,y_k), A(x_k,Tx), A(y_k,Ty), A(x_k,Ty), A(y_k,Tx)\Big\}.$$

(3) (*The condition of* $(m+r)$*-polygon*). There exists a constant $\alpha > 1$ such that for all $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ in X^k the following inequality holds in the form as

$$A(Tx,Ty) \geqslant \alpha \operatorname{trstdiam}\Big\{x_k, y_k, Tx, Ty, \dots, T^m x, T^r y\Big\}$$

for arbitrary fixed integers $m, r \geqslant 0$. (*This is a linear condition for trs.diameter of finite number of points*).

(4) There exists a nondecreasing function $\psi : [0,+\infty] \to [0,+\infty]$ satisfying the following condition in the form $\liminf_{z\to t-0} \psi(z) > t$ for every $t \in \mathbb{R}^0_+$ such that

$$A(Tx,Ty) \geqslant \psi\Big(\operatorname{trstdiam}\{x_k, y_k, Tx, Ty, \dots, T^m x, T^r y\}\Big)$$

for arbitrary integers $m, r \geqslant 0$ and for all $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ in X^k. (*This is a nonlinear condition for trs.diameter of finite number of points*).

(5) There exists an increasing mapping $f : [0,+\infty]^5 \to [0,+\infty]$ satisfying the following condition in the form $\lim_{z\to t-0} f(z,z,z,z,z) > t$ for every $t \in \mathbb{R}^0_+$ such that

$$A(Tx,Ty) \geqslant f\Big(A(x_k,y_k), A(x_k,Tx), A(y_k,Ty), A(x_k,Ty), A(y_k,Tx)\Big)$$

for all $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ in X^k.

In connection with the preceding facts, we are now in a position to formulate a localization of Theorem 12 in the following form.

Theorem 14. *Let $X := (X, A)$ be a lower spring orbitally complete lower spring transversal space, and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Suppose that there exists a mapping $\varphi : [0,+\infty] \to [0,+\infty]$ satisfying* (Id) *such that*

$$\operatorname{trstdiam}\{Tx, T^2x, \dots\} \geqslant \varphi\Big(\operatorname{trstdiam}\Big\{x_k, Tx, T^2x, \dots\Big\}\Big)$$

for every $x = (x_1,\dots,x_k)$ in X^k. If $t \mapsto \operatorname{trstdiam}\mathcal{O}(t, T(t,\dots,t))$ or $t \mapsto A(t, T(t,\dots,t))$ is T-orbitally upper semicontinuous, then there exists at least one solution of the equation (Ea).

The proof of this localization statement is totally analogous with the preceding proof of Theorem 12. Thus the proof of this result we omit.

For further facts which are connected with the preceding problems of fixed point on cartesian product of spaces see:

Avramescu [1970], Berinde [1992], Caius [1991], Dezsö [2005], Kwapisz [1979], Matkowski [1973], Mureşan [1988], Nicolescu [1975], Pascali [1975], Pascali-Zilli [1978], Petruşel [1984], Rus [1979], Rus [1981], Şerban [2000], Turinici [1980], Tasković [1973], Tasković [1975], Van der Walt [1963], Wazewski [1960], Ginsberg [1954], Brown [1974], Brown [1982], Singh-Gairola [1991], and Dold [1986].

Historical facts. By R. Brown[1] in 1974 year Kazimierz Kuratowski asked a question that puzzled topologists for a long time. This is the story of Kuratowski's question.

The Question and the Answer. A topological space has the fixed point property (fpp) if for any map (=continuous function) $f : X \to X$ there exists a fixed point, that is, a point $x \in X$ such that $f(x) = x$. Kuratowski's question was of the form: if space X and Y have the fpp, does their cartesian product $X \times Y$ have the fpp? (Recall that, as a set, $X \times Y$ consists of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. The open sets in $X \times Y$ are the unions of cartesian products of open sets of X and of Y. In most of this part, the topological setup is much simpler. Both X and Y will be subsets of euclidean spaces, so $X \times Y$ will just inherit its topology from a higher-dimensional eucledean space.)

At the risk of ruining all the suspense, I'll tell you right away that the answer to Kuratowski's question is "no", even if X and Y are required to be very well-behaved spaces.

Seeing the Answer. I will point out in the next part that we are discussing an old and rather basic problem, but one whose satisfactory resolution is quite recent. Although the complete solution involves some pretty sophisticated topology, you need remarkably little background information in order to understand the main points of the solution. The fact that the answer is negative helps explain why this should be so: the problem is solved by explicitly constructing spaces X and Y with the fpp such that $X \times Y$ lacks the fpp.

At times I will need to base my claims on advanced topics in topology. Even at these points it turns out to be easy to describe precisely what I'm using and to refer the curious reader to the relevant literature.

Kuratowski's Question. The published history of our problem began in 1930 when Kuratowski asked: If X and Y are peano continua with the fpp, does $X \times Y$ have the fpp? (A *peano continuum* is a compact, connected, and locally connected metric space.)

I emphasize "published" history because I'm quite certain the problem was old and well known among topologists long before Kuratowski published it. Such a

[1] Robert F. Brown received his Ph. D. from the University of Wisconsin where he was a student of Edward Fadell. He has since been at UCLA except for a visiting Professorship at the University of Arizona and two sabbatical years at the University of Warwick - England. His research area is algebraic topology, with a particular interest in fixed point theory.

formal presentation usually indicates that a problem has been around long enough and has been discussed sufficiently so that the proposer is sure it is both difficult and of some significance to its subject area.

Furthermore, notice, that Kuratowski put hypothesis-peano-continuum on his spaces. This restriction suggests to me that enough was known about the question so that wildly pathological counterexamples has been discovered (though none seems to have been published at that time) or at least that the existence of such of such examples was suspected.

Motivation. Turning from the question of when the problem arose, let's ask a more important one: why would topologists be interested in it? Kuratowski didn't include any motivation for studying the problem, but I can suggest two reasons the problem came up.

The first reason is a concern with what might be called "the foundations of topology". The classical definition of topology is: the study of properties invariant under homeomorphisms. The fpp is such a property. What does it mean to "study" a property? The answer to that question would be very long, but certainly a part of the answer is: find out whether the property is preserved under the various basic constructions of topology – such as forming cartesian products. The most famous positive result, Tychonoff's theorem, solves the cartesian product problem for the property of compactness. On the other hand, it has long been known that the cartesian product of normal spaces is not necessarily normal. Thus Kuratowski's question was, and is, interesting because its solution tells us something about the very nature of topology.

The second reason for being interested in the behavior of the fpp under cartesian products is tied up with the early history of topology. Let $\mathbb{R}^n$ denote euclidean n-dimensional space and let I^n be the standard n-cell.

It is easy to show that $I = I^1$ has the fpp. The Brouwer Fixed Point Theorem states that I^n has the fpp for all n. Since I^n is the cartesian product of n copies of I, the Brouwer Theorem suggests that the fpp might behave well under the cartesian product construction. Moreover, although nowadays the Brouwer theorem is easy to prove using elementary algebraic topology, back in the 1920's the subject was much less well-developed and the existing proofs of the theorem were pretty difficult. But suppose the fpp *were* preserved under cartesian products. Then the Brouwer Fixed Point Theorem would be in immediate consequence: prove the easy $n = 1$ case and apply induction.

A Pathological Example. It may be that Kuratowski's publication of the cartesian product problem for the fpp tended to focus topologists' attention on the problem. If it did, they don't seem to have had much success because there is no mention of the problem in the literature until many years later. A very special affirmative answer in 1956 by Eldon Dyer (for "chainable continua") is probably best viewed as a generalization of the Brouwer Fixed Point Theorem.

The first significant contribution to the solution of Kuratowski's problem was the work of Edwin Connell. In 1959 he gave the first published example to demonstrate that without some restriction, such as Kuratowski's to peano continua, the fpp would not be preserved under cartesian product.

The example uses the subset X of $\mathbb{R}^2$ consisting of points $(x, \sin \pi/(1-x))$ for $0 \leqslant x < 1$ and the point $(1, 1)$ pictured in Fig. 1. It is easy to prove that X has the fpp. Connell then shows that $X \times X$ does not have the fpp. I won't repeat the argument, but I'll need to refer later to the method of proof. Connell proves that in any metric space with the fpp, "every locally finite chain of arcs in finite". (For the point I wont to make, it doesn't matter what these words mean.) Then he constructs in $X \times X$ an infinite, locally finite chain of arcs. Thus, although all that is required to prove that a space lacks the fpp is to exhibit a single self-map of it without fixed points, it seems such a direct proof was not available for $X \times X$.

Kuratowski's problem was still unsolved because X is not a peano continuum – it is neither compact nor locally connected. But whether or not Kuratowski had been guided by a pathological example in stating the problem in 1930, such an example was now available.

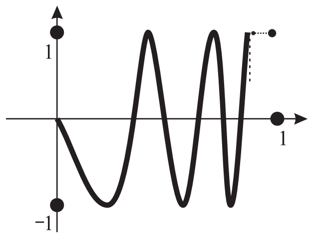

FIGURE 1.

Kuratowski's Question Answered. After 1959, other examples of spaces X and Y with the fpp such that $X \times Y$ lacks the fpp were published, but in neither case were both X and Y peano continua. Then, in 1967, Edward Fadell and his student William Lopez presented an example of a peano continuum (in fact a finite polyhedron) X with the fpp such that $X \times I$ doesn't have fpp. Thus Kuratowski's question was answered at last. I will describe such a polyhedron X in further. It's not the original Fadell-Lopez example, but instead a somewhat simpler example suggested by Glen Bredon.

A Wedge is not Locally Euclidean. You have to known a little about these examples in order to understand what happened after 1967. The Fadell-Lopez type of example is a "wedge" of two sets. If X is a space (think of it as a subset of a euclidian space because that's what we'll be concerned with), A and B are closed subsets of X such that $X = A \cup B$ and $A \cap B = \{x_0\}$, a single point, then X is called the *wedge* of A and B. Write $X = A \vee B$. Notice that $X \backslash \{x_0\}$ is disconnected.

The important property of a wedge for the purpose of fixed point theory is this easily proved fact: if A and B each have the fpp, so does $X = A \vee B$.

A space X is *locally n-euclidean* at a point x if there is a neighborhood U of x in X homeomorphic to $\mathbb{R}^n$. If a connected space X is locally n-euclidean at x, for $n \geqslant 2$, then $X \backslash \{x\}$ is still connected. So if $X = A \vee B$ and $\{x_0\} = A \cap B$ then, since $X \backslash \{x_0\}$ is disconnected, X certainly can't be locally n-euclidean at x_0, $n \geqslant 2$. There is shows just how very noneuclidean $X = A \vee B$ is at x_0, even when (as in this example), A and B are both locally euclidean at x_0.

The Manifold Problem. The Fadell-Lopez type of example is of the form $X = A \vee B$ where A and B are peano continua with the fpp, so X is a peano continuum and, as I pointed out above, it follows that X must also have the fpp. The observations that (1) his example depended on the wedge structure to prove X has the fpp and (2) a wedge is strikingly noneuclidean, at least at one point, led Fadell to wonder if the key to the example lay in the lack of locally euclidean structure. So, in 1970, Fadell asked Kuratowski's question in this stronger form: if X and Y are compact manifolds with the fpp, does $X \times Y$ have the fpp? (An n- *manifold* is a metric space that is locally n-euclidean at every point.)

Almost immediately after Fadell raised the question, Bredon produced example of nice spaces X and Y with the fpp such that $X \times Y$ does not have the fpp, and neither X nor Y is a wedge. But Bredon's spaces are not manifolds according to the definition I just quoted. In 1977, Sufian Husseini extended Bredon's ideas to sonstruct examples that are manifolds, so the answer to Kuratowski's question is still "no" even if we require spaces as nice as these.

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MILAN R. TASKOVIĆ
FACULTY OF MATHEMATICS
11000 BELGRADE, P.O. BOX 550
SERBIA

Home Address:
MILAN R. TASKOVIĆ
NEHRUOVA 236
11070 BELGRADE
SERBIA
E-mail address: `andreja@predrag.us`