Mathematica Moravica
Vol. **14 - 1** (2010), 47–51

Some Equalities Which Hold in the (n,m)-Group (Q,A) for $n \geqslant 2m$

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Abstract. In this paper, we have proved two equalities which hold in an (n,m)-group (Q,A) for $n \geqslant 2m$. The first of them is a generalization of the equality $(a\cdot b)^{-1} = b^{-1}\cdot a^{-1}$, which holds in the binary group $(Q,\cdot)$. The second of them is equality
$A(x_1^m, b_1^{n-2m}, y_1^m) = A(A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, e(b_1^{n-2m}))^{-1}), a_1^{n-2m}, y_1^m)$.

1. Preliminaries

The notion of an (n,m)-group was introduced by Čupona in 1983, as a generalization of the notion of an n-group, that is of a group. Let Q be a nonempty set and let A be a mapping of the set Q^n into the set Q^m. Then, we say that (Q,A) is an (n,m)-groupoid.

Definition 1.1 ([1])**.** Let $n \geqslant m+1$ and let (Q,A) be an (n,m)-groupoid. We say that (Q,A) is an (n,m)-group iff the following statements hold:

(i) For every $x_1^{2n-m} \in Q$ and for every $i,j \in \{1,\ldots,n-m+1\}$, $i<j$, the following law holds

$$A\left(x_1^{i-1}, A\left(x_i^{i+n-1}\right), x_{i+n}^{2n-m}\right) = A\left(x_1^{j-1}, A\left(x_j^{j+n-1}\right), x_{j+n}^{2n-m}\right). \tag{1}$$

(ii) For every $i \in \{1,\ldots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A\left(a_1^{i-1}, x_1^m, a_i^{n-m}\right) = a_{n-m+1}^n. \tag{2}$$

Remark. The equality (1) is called an $\langle i,j\rangle$-associative law. If in the (n,m)-groupoid (Q,A) the statement (i) holds, then we say that (Q,A) is an (n,m)-semigroup.

An important notion in the theory of the (n,m)-group is a $\{1,n-m+1\}$-neutral operation. It is a generalization of the notion of a neutral element in binary structures. The notion of a $\{1,n-m+1\}$-neutral operation was introduced by Ušan in 1989.

2000 *Mathematics Subject Classification.* Primary: 20N15.
Key words and phrases. (n,m)-group, $\{1,n-m+1\}$-neutral operation, inverse operation of the (n,m)-groupoid.

Definition 1.2 ([2]). Let $n \geqslant 2m$ and let (Q,A) be an (n,m)-groupoid. Also, let e_L, e_R and e be mappings of the set Q^{n-2m} into the set Q^m. Then:

(i) e_L is a left $\{1, n-m+1\}$-neutral operation of the (n,m)-groupoid (Q,A) iff for every $x_1^m \in Q$ and for every $a_1^{n-2m} \in Q$ the following equality holds
$$A\left(e_L\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m;$$

(ii) e_R is a right $\{1, n-m+1\}$-neutral operation of the (n,m)-groupoid (Q,A) iff for every $x_1^m \in Q$ and for every $a_1^{n-2m} \in Q$ the following equality holds
$$A\left(x_1^m, a_1^{n-2m}, e_R\left(a_1^{n-2m}\right)\right) = x_1^m;$$

(iii) e is a $\{1, n-m+1\}$-neutral operation of the (n,m)-groupoid (Q,A) iff for every $x_1^m \in Q$ and for every $a_1^{n-2m} \in Q$ the following equalities hold
$$A\left(e\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m$$
and
$$A\left(x_1^m, a_1^{n-2m}, e\left(a_1^{n-2m}\right)\right) = x_1^m.$$

Remark. For $(n,m) = (2,1)$, the Definition 1.2 is a definition of a neutral element in the binary groupoid (Q,A).

2. Auxiliary Propositions

Some properties of the $\{1, n-m+1\}$-neutral operation, which we used in main theorems, are contained in the following propositions.

Proposition 2.1 ([2]). *Every (n,m)-group (Q,A), $n \geqslant 2m$, has a $\{1, n-m+1\}$-neutral operation.*

Proposition 2.2 ([4]). *Let (Q,A) be an (n,m)-groupoid and $n \geqslant 2m$. Further on, let the following statements hold:*

(i) *The $\langle 1, n-m+1\rangle$-associative law holds in (Q,A);*

(ii) *For every $a_1^n \in Q$ there is at least one $x_1^m \in Q^m$ such that the equality $A\left(a_1^{n-m}, x_1^m\right) = a_{n-m+1}^n$ holds;*

(iii) *For every $a_1^n \in Q$ there is at least one $y_1^m \in Q^m$ such that the equality $A\left(y_1^m, a_1^{n-m}\right) = a_{n-m+1}^n$ holds.*

Then there are mappings $e : Q^{n-2m} \to Q^m$ and ${}^{-1} : Q^{n-m} \to Q^m$ such that the following laws

(a) $A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, x_1^m\right)\right) = x_1^m,$

(b) $A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = x_1^m,$

(c) $A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = e\left(a_1^{n-2m}\right),$

(d) $A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right) = e\left(a_1^{n-2m}\right)$

hold in the algebra $\left(Q; A,^{-1}, e\right)$.

Remark. The mapping $^{-1} : Q^{n-m} \to Q^m$ can be defined by the neutral operation in the following way: let $E : Q^{2n-3m} \to Q^m$ be a $\{1, 2n-2m+1\}$-neutral operation of the $(2n-m, m)$-groupoid $\left(Q, \overset{2}{A}\right)$, where is

$$\overset{2}{A}\left(x_1^{2n-m}\right) \overset{def}{=} A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right).$$

Then the mapping $^{-1} : Q^{n-m} \to Q^m$ is determined by the following equality: $\left(a_1^{n-2m}, b_1^m\right)^{-1} \overset{def}{=} E\left(a_1^{n-2m}, b_1^m, a_1^{n-2m}\right)$ and called an inverse operation of the (n, m)-groupoid (Q, A).

3. Main Results

In the following theorem, a well known equality $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$, which holds in binary structures, is generalized.

Theorem 3.1. *Let* (Q, A) *be an* (n, m)*-group,* $n \geqslant 2m$ *and let* $^{-1} : Q^{n-m} \to Q^m$ *be its inverse operation. Then, for every* $a_1^m, b_1^m, c_1^{n-2m} \in Q$ *the following equality holds*

$$\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1} = A\left(\left(c_1^{n-2m}, b_1^m\right)^{-1}, c_1^{n-2m}, \left(c_1^{n-2m}, a_1^m\right)^{-1}\right).$$

Proof. Let e be a $\{1, n-m+1\}$-neutral operation of the (n, m)-group (Q, A) (it exists by proposition 2.1) and let $a_1^m, b_1^m, c_1^{n-2m} \in Q$ be arbitrary elements of the set Q. Then the following equivalence holds:

$$A\left(A\left(a_1^m, c_1^{n-2m}, b_1^m\right), c_1^{n-2m}, x_1^m\right) = e\left(c_1^{n-2m}\right) \overset{1.1(ii)}{\Leftrightarrow}$$

$$A\left(\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1}, c_1^{n-2m}, A\left(A\left(a_1^m, c_1^{n-2m}, b_1^m\right), c_1^{n-2m}, x_1^m\right)\right) =$$

$$A\left(\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1}, c_1^{n-2m}, e\left(c_1^{n-2m}\right)\right)$$

By the proposition 2.2 (a), for $c_1^{n-2m} \equiv a_1^{n-2m}$ and $A\left(a_1^m, c_1^{n-2m}, b_1^m\right) \equiv b_1^m$, we conclude that the following equality holds:

$$A\left(\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1}, c_1^{n-2m},\right.$$
$$\left.A\left(A\left(a_1^m, c_1^{n-2m}, b_1^m\right), c_1^{n-2m}, x_1^m\right)\right) = x_1^m.$$

By the definition of a $\{1, n-m+1\}$-neutral operation, we conclude that the following equality holds:

$$A\left(\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1}, c_1^{n-2m}, e\left(c_1^{n-2m}\right)\right) =$$
$$= \left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1}.$$

Hence, we have the following equality:

$$x_1^m = \left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1} \tag{3}$$

Furthermore, the following sequence of equivalences holds:

$$A\left(A\left(a_1^m, c_1^{n-2m}, b_1^m\right), c_1^{n-2m}, x_1^m\right) = e\left(c_1^{n-2m}\right) \overset{1.1(i)}{\Leftrightarrow}$$
$$A\left(a_1^m, c_1^{n-2m}, A\left(b_1^m, c_1^{n-2m}, x_1^m\right)\right) = e\left(c_1^{n-2m}\right) \overset{1.1(ii)}{\Leftrightarrow}$$
$$A\left(\left(c_1^{n-2m}, a_1^m\right)^{-1}, c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, A\left(b_1^m, c_1^{n-2m}, x_1^m\right)\right)\right) =$$
$$= A\left(\left(c_1^{n-2m}, a_1^m\right)^{-1}, c_1^{n-2m}, e\left(c_1^{n-2m}\right)\right)$$

By the proposition 2.2 (a) we conclude that the following equality holds:

$$A\left(\left(c_1^{n-2m}, a_1^m\right)^{-1}, c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, A\left(b_1^m, c_1^{n-2m}, x_1^m\right)\right)\right) =$$
$$A\left(b_1^m, c_1^{n-2m}, x_1^m\right).$$

By the definition of a $\{1, n-m+1\}$-neutral operation, we conclude that the following equality holds:

$$A\left(\left(c_1^{n-2m}, a_1^m\right)^{-1}, c_1^{n-2m}, e\left(c_1^{n-2m}\right)\right) = \left(c_1^{n-2m}, a_1^m\right)^{-1}.$$

Hence, we have the following equality:

$$A\left(b_1^m, c_1^{n-2m}, x_1^m\right) = \left(c_1^{n-2m}, a_1^m\right)^{-1}.$$

Further on, the following equivalence holds:

$$\begin{aligned}(4)\qquad & A\left(b_1^m, c_1^{n-2m}, x_1^m\right) = \left(c_1^{n-2m}, a_1^m\right)^{-1} \overset{1.1(ii)}{\Leftrightarrow}\\ & A\left(\left(c_1^{n-2m}, b_1^m\right)^{-1}, c_1^{n-2m}, A\left(b_1^m, c_1^{n-2m}, x_1^m\right)\right) =\\ & = A\left(\left(c_1^{n-2m}, b_1^m\right)^{-1}, c_1^{n-2m}, \left(c_1^{n-2m}, a_1^m\right)^{-1}\right) \overset{2.2(a)}{\Leftrightarrow}\\ & x_1^m = A\left(\left(c_1^{n-2m}, b_1^m\right)^{-1}, c_1^{n-2m}, \left(c_1^{n-2m}, a_1^m\right)^{-1}\right).\end{aligned}$$

By (3) and (4) we conclude that the following equality holds:

$$\left(c_1^{n-2m}, A\left(a_1^m, c_1^{n-2m}, b_1^m\right)\right)^{-1} =$$
$$= A\left(\left(c_1^{n-2m}, b_1^m\right)^{-1}, c_1^{n-2m}, \left(c_1^{n-2m}, a_1^m\right)^{-1}\right). \qquad \square$$

Theorem 3.2. *Let (Q,A) be an (n,m)-group, $n \geqslant 2m$, let $e : Q^{n-2m} \to Q^m$ be its $\{1, n-m+1\}$-neutral operation and let ${}^{-1} : Q^{n-m} \to Q^m$ be its inverse operation. Then for every $a_1^{n-2m}, b_1^{n-2m}, x_1^m, y_1^m \in Q$ the following equality holds:*

$$\begin{aligned}(5)\qquad & A\left(x_1^m, b_1^{n-2m}, y_1^m\right) =\\ & = A\left(A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right)^{-1}\right), a_1^{n-2m}, y_1^m\right).\end{aligned}$$

Proof. For $n = 2m$ the equality (5) reduces to the equality: $A\left(x_1^m, y_1^m\right) = A\left(x_1^m, y_1^m\right)$.

Let $n > 2m$. For the arbitrary sequences $a_1^{n-2m}, b_1^{n-2m}, x_1^m, y_1^m, z_1^m \in Q$ the following sequence of equalities holds:

$$A\left(z_1^m, a_1^{n-2m}, y_1^m\right) \overset{1.2}{=} A\left(z_1^m, a_1^{n-2m}, A\left(e\left(b_1^{n-2m}\right), b_1^{n-2m}, y_1^m\right)\right) \overset{1.1(i)}{=}$$
$$= A\left(A\left(z_1^m, a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right), b_1^{n-2m}, y_1^m\right)$$

Hence, we have the equality:

$$(6) \qquad A\left(A\left(z_1^m, a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right), b_1^{n-2m}, y_1^m\right) = A\left(z_1^m, a_1^{n-2m}, y_1^m\right).$$

Further, the following sequence of equivalences holds:

$$A\left(z_1^m, a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right) = x_1^m \overset{1.1(ii)}{\Leftrightarrow}$$
$$A\left(A\left(z_1^m, a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right), a_1^{n-2m}, \left(a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right)^{-1}\right) =$$
$$= A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right)^{-1}\right) \overset{2.2(b)}{\Leftrightarrow}$$
$$z_1^m = A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right)^{-1}\right).$$

If the first equality of the above sequence of equivalences is put to the left side of equality (6) and the third equality to the right side of equality (6), then the statement of this theorem is proved. □

The above discussed equality for $m = 1$, was proved in [3].

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