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The Main Eigenvalues of the Seidel Matrix

Houqing Zhou*

Abstract. Let G be a simple graph with vertex set $V(G)$ and $(0,1)$-adjacency matrix A. As usual, $A^*(G) = J - I - 2A$ denotes the Seidel matrix of the graph G. The eigenvalue λ of A is said to be a main eigenvalue of G if the eigenspace $\varepsilon(\lambda)$ is not orthogonal to the all-1 vector $\mathbf{e}$. In this paper, relations between the main eigenvalues and associated eigenvectors of adjacency matrix and Seidel matrix of a graph are investigated.

1. Introduction

Let G be a simple graph with n vertices. We write $V(G)$ for the vertex set of G, and $E(G)$ for the edge set of G. The spectrum of the graph G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its $(0,1)$ adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0,-1,1)$ adjacency matrix $A^* = A^*(G)$ and its denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively.

For a real symmetric matrix A, an eigenvalue of A is called *simple* if its algebraic multiplicity is one, and the eigenvalue λ of A is said to be a main eigenvalue of G if the eigenspace $\varepsilon(\lambda)$ is not orthogonal to the all-1 vector $\mathbf{e}$. Any real symmetric matrix A has at least one main eigenvalue. Furthermore, matrix A has exactly one main eigenvalue if and only if the vector $\mathbf{e} = (1,1,\ldots,1)^T$ is an eigenvector of A. For a graph G, its main eigenvalues are those of $A(G)$, and G has exactly one main eigenvalue if and only if G is a regular graph. There are many results and their applications on the main eigenvalues of graphs, see[1],[2],[3],[4],[6],[7], but it is still an open problem to characterize the graphs with exactly $l (l \geq 3)$ main eigenvalues(as the case $l = 2$ has been settled, see [4],[5]). It is well known that if the graph

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G is r-regular graph, in other words, G has exactly one main eigenvalue, then([1] p. 30)

$$P_G^*(\lambda) = (-1)^n 2^n \frac{\lambda + 1 + 2r - n}{\lambda + 1 + 2r} P_G\left(-\frac{\lambda + 1}{2}\right). \tag{1.1}$$

Hence the Seidel spectrum of regular graph is determined by its adjacency spectrum.

The aim of this paper is to prove that the main Seidel eigenvalues of a graph are recoverable by the main eigenvalues of adjacency matrix and associated eigenvectors and give a method for computing the main Seidel eigenvalues in terms of the main eigenvalues and associated eigenvectors of adjacency matrix.

The rest of the paper is organized as follows. In Section 2 contains some definitions. In Section 3 we will describe the relation between main eigenvalues of G and main eigenvalues of $A^*(G)$, and prove several theorems on the main eigenvalue of graphs.

2. Some basic notions

Let A have spectral decomposition

$$A = \mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_m P_m. \tag{2.1}$$

The main angles of G are the numbers $\beta_1, \beta_2, \ldots, \beta_m$, where $\beta_i = \frac{1}{\sqrt{n}}\|P_i \mathbf{e}\|$ $(i = 1, 2, \ldots, m)$. These are the cosines of the angles between $\mathbf{e}$ and the eigenspaces of A, and so μ_i is a main eigenvalue if and only if $\beta_i \neq 0$. Since $\|\mathbf{e}\|^2 = \sum_{i=1}^m \|P_i \mathbf{e}\|^2$, we have $\sum_{i=1}^m \beta_i^2 = 1$. The main eigenvalues include the index (largest eigenvalue) of G because there exists a corresponding eigenvector with no negative entries, see [1].

We take the main eigenvalues of G to be $\mu_1, \mu_2, \ldots, \mu_s$, with μ_1 the index of G; no further ordering is assumed for $\mu_2, \mu_3, \ldots, \mu_s$.

First, we introduce some notation and preliminaries which will be useful to obtain the main results.

It is not difficult to see the following lemmas:

Lemma 2.1. *(see* [1]*) The relation between the characteristic polynomial $P_G(\lambda)$ of a graph G and the characteristic polynomial $P_G^*(\lambda)$ of the Seidel adjacency matrix $A^*(G)$ of G can be written in the form*

$$P_G(\lambda) = \frac{(-1)^n}{2^n} \cdot \frac{P_G^*(-2\lambda - 1)}{1 + \frac{1}{2\lambda} H_G(\frac{1}{\lambda})}. \tag{2.2}$$

Lemma 2.2. *(see* [9]*) If N_k denotes the number of walks of length k in G, then*

$$N_k = n \sum_{i=1}^{s} \mu_i^k \beta_i^2.$$

According to [1], the *walk generating function* $H_G(t)$ is defined by $H_G(t) = \sum_{k=0}^{+\infty} N_k t^k$, and it follows from Lemma 2.2 that

$$H_G(t) = \sum_{i=1}^{s} \frac{n\beta_i^2}{1-\mu_i t}. \tag{2.3}$$

Using above lemma, we see that the main eigenvalues of $A^*(G)$ are determined by the main eigenvalues of G.

3. MAIN RESULTS

We proceed now to the investigation of the main Seidel eigenvalues of a graph G. We shall apply above lemma and a result from [1], the following result is immediately obtained.

Theorem 3.1.

$$P_G^*(\lambda) = (-2)^n P_G(-\frac{\lambda+1}{2})(1 - n\sum_{i=1}^{s} \frac{\beta_i^2}{\lambda+1+2\mu_i}). \tag{3.1}$$

Proof. According to (2.2) and (2.3), by a straightforward calculation, hence we have (3.1). □

Note that $A^*(G) = J - I - 2A(G)$, where the symbol J denotes a square matrix all of whose entries are equal to 1, I means a unit matrix in general, respectively. If α is an eigenvector of $A(G)$ with eigenvalue μ such that $\mathbf{e}^T\alpha = 0$, then α is also an eigenvector of $A^*(G)$ with eigenvalue $-1-2\mu$, since $A^*(G)\alpha = (J - I - 2A(G))\alpha = J\alpha - \alpha - 2A(G)\alpha = (-1-2\mu)\alpha$. In other words, the non-main eigenvalues of $A^*(G)$ are determined by those of $A(G)$. Using this fact, we can simplify Equation (3.1) so that it involves only the main eigenvalues $\mu_1, \mu_2, \ldots, \mu_s$ and $\lambda_1^*, \lambda_2^*, \ldots, \lambda_s^*$ of $A(G)$ and $A^*(G)$, respectively, i.e.

$$\prod_{i=1}^{s}(\lambda - \lambda_i^*) = \prod_{i=1}^{s}(\lambda + 1 + 2\mu_i)(1 - n\sum_{i=1}^{s} \frac{\beta_i^2}{\lambda+1+2\mu_i}). \tag{3.2}$$

Using Equation (3.2) for both $A(G)$ and $A^*(G)$, we can see the main eigenvalues of $A^*(G)$ are determined by the main eigenvalues and corresponding eigenvector of $A(G)$. But we can say more.

Theorem 3.2. *Suppose that μ_k is a main eigenvalue of $A(G)$, then $-1-2\mu_k$ cannot be a main eigenvalue of $A^*(G)$.*

Proof. By evaluating Equation (3.1) at $-1-2\mu_k$, we have

$$\prod_{i=1}^{s}(-1-2\mu_k - \lambda_i^*) = 2^s \prod_{i=1, i\neq k}^{s}(\mu_i - \mu_k)(1 - n\sum_{i=1}^{s} \frac{\beta_i^2}{2(\mu_i - \mu_k)}).$$

Hence for $i = 1, 2, \ldots, s$, $\lambda_i^* \neq -1-2\mu_k$. □

For example, if G is the cycle C_4 then its the main eigenvalue is 2, via calculation, show that -5 is non-main eigenvalue of $A^*(G)$, its main eigenvalue is 3.

A consequence of this theorem is the following.

Corollary 3.3. *Suppose that μ is a simple main eigenvalue of $A(G)$. Then $-1-2\mu \notin \sigma^*(G)$.*

Now, we give the following lemma.

Lemma 3.4. *Let $\mu \in \sigma(G)$. Then $-1-2\mu \in \sigma^*(G)$ if and only if $\mathbf{e}^T\alpha = 0$ for some eigenvector α corresponding to the eigenvalue μ of $A(G)$.*

Proof. Sufficiency follows from the Theorem 3.1.

To prove necessity. Assume that $-1-2\mu \in \sigma^*(G)$ and note that μ cannot be simple main eigenvalue of $A(G)$ by Corollary 3.3. By Theorem 3.2, $-1-2\mu$ is not a main eigenvalue of $A^*(G)$. Thus $A^*(G)$ has an eigenvector α corresponding to $-1-2\mu$ such that $\mathbf{e}^T\alpha = 0$ and α is also an eigenvector of $A(G)$ corresponding to μ. □

Next, we present the main result of this note that the main eigenvalues and associated eigenvectors of $A^*(G)$ are recoverable from those of $A(G)$.

Theorem 3.5. *Let $\mu_1, \mu_2, \ldots, \mu_s$ be the main eigenvalues of the graph G, and let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be corresponding orthonormal eigenvectors. Let E be the $s \times s$ matrix whose (i,j)-entry is $\mathbf{e}^T\alpha_i\mathbf{e}^T\alpha_j$, and let $M = E - I - 2diag(\mu_1, \mu_2, \ldots, \mu_s)$. Then eigenvalues of M are precisely the main eigenvalues of $A^*(G)$. Moreover, if $c = (c_1, c_2, \ldots, c_s)^T$ is an eigenvector of M corresponding to the eigenvalue λ^*, then $\sum_{i=1}^{s} c_i\alpha_i$ is an eigenvector of $A^*(G)$ corresponding to λ^*.*

Proof. Let λ^* be a main eigenvalue of $A^*(G)$ with the corresponding eigenvector α^*. Since any eigenvector α of $A(G)$ such that $\mathbf{e}^T\alpha = 0$ is also an eigenvector of $A^*(G)$ and vice versa, two spaces spanned by the eigenvectors of $A(G)$ and $A^*(G)$ the sum of whose entries is zero are identical. Equivalently, the eigenvectors associated with the main eigenvalues of $A^*(G)$ span the same space as that of $A(G)$. Thus we can express α^* as a linear combination of eigenvectors $\alpha_1, \alpha_2, \ldots, \alpha_s$, $\alpha^* = \sum_{i=1}^{s} c_i\alpha_i$. Hence $A(G)\alpha^* = \sum_{i=1}^{s} c_i\mu_i\alpha_i$. As $A(G) = \frac{1}{2}(J - I - A^*(G))$, so $A(G)\alpha^* = \frac{1}{2}(J - \alpha^* - \lambda^*\alpha^*)$. Thus $J\alpha^* = 2A(G)\lambda^* + (1+\lambda^*)\alpha^*$. Combining above two expressions we get

$$(\mathbf{e}^T\alpha^*)\mathbf{e} = (\mathbf{e}^T\mathbf{e})\alpha^* = J\alpha^* = \sum_{i=1}^{s} c_i(2\mu_i + 1 + \lambda^*).$$

Taking the scalar product of both side with $\alpha_i, i = 1, 2, \ldots, s$. We obtain

$$\mathbf{e}^T\alpha^*\mathbf{e}^*\alpha_j = \sum_{i=1}^{s} c_i\mathbf{e}^T\alpha_i\mathbf{e}^T\alpha_j = (2\mu_j + 1 + \lambda^*)c_j. \tag{3.3}$$

In matrix form, the set of equations represented by (7) is

$$(E - I - 2diag(\mu_1, \mu_2, \dots, \mu_s))c = \lambda^* c.$$

Thus λ^* is an eigenvalue of M with corresponding eigenvector c, and Theorem follows. □

Similarly, the main eigenvalues and associated eigenvectors of $A(G)$ are recoverable from those of $A^*(G)$.

Theorem 3.6. *Let $\lambda_1^*, \lambda_2^*, \dots, \lambda_l^*$ be the main eigenvalues of $A^*(G)$ and $\alpha_1^*, \alpha_2^*, \dots, \alpha_l^*$ be the associated orthonormal eigenvectors. Let E be the $l \times l$ matrix whose (i,j)-entry is $\mathbf{e}^T\alpha_i^*\mathbf{e}^T\alpha_j^*$, and $M^* = \frac{1}{2}(E-I-diag(\lambda_1^*, \lambda_2^*, \dots, \lambda_l^*))$. Then eigenvalues of M^* are precisely the main eigenvalues of $A(G)$. Further more, if $b = (b_1^*, b_2^*, \dots, b_l^*)^T$ is an eigenvector that corresponding to an eigenvalue μ^* of M^*, then $\sum_{j=1}^{l} b_j^*\alpha_j^*$ is an eigenvector of $A(G)$ corresponding to μ^*.*

From Equation (1) we have $2\lambda_1 + \lambda_1^* = n - 1$ for regular graph. The following is a generalization of this fact.

Corollary 3.7. *Let $\lambda_1, \lambda_2, \dots, \lambda_l$ and $\lambda_1^*, \lambda_2^*, \dots, \lambda_l^*$ are all main eigenvalues of $A(G)$ and $A^*(G)$, respectively. Then*

$$\sum_{i=1}^{l}(2\lambda_i + \lambda_i^*) = n - l.$$

Proof. Since $\lambda_1^*, \lambda_2^*, \dots, \lambda_l^*$ are all eigenvalues of matrix M in Theorem 5, we get

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i^* = trace(M) &= \sum_{i=1}^{l}(\mathbf{e}^T\alpha_i)^2 - l - \sum_{i=1}^{l}2\lambda_i \\ &= \sum_{i=1}^{l}n_i - l - \sum_{i=1}^{l}2\lambda_i \\ &= n - l - \sum_{i=1}^{l}2\lambda_i\end{aligned}$$

Hence Corollary follows. □

From Theorem 3.5. we know that if $A(G)$ has few main eigenvalues then the main eigenvalues of $A^*(G)$ can be obtained easily. The following is an example.

Example 3.8. Let $G = G_1 \bigcup G_2$ be the union of two regular graphs G_1 and G_2 of order n_1 and n_2 and degree r_1 and $r_2 (r_1 \neq r_2)$, respectively. It is easy to see that $A(G)$ has exactly two main eigenvalues r_1 and r_2 and associated orthonormal eigenvector are $\alpha_1 = \frac{1}{\sqrt{n_1}}(\overbrace{1, 1, \ldots, 1}^{n_1}, 0, \ldots, 0)^T$ and $\alpha_2 = \frac{1}{\sqrt{n_2}}(0, \ldots, 0, \overbrace{1, 1, \ldots, 1}^{n_2})^T$. Thus

$$M = \begin{bmatrix} n_1 - 1 - 2r_1 & \sqrt{n_1 n_2} \\ \sqrt{n_1 n_2} & n_2 - 1 - 2r_2 \end{bmatrix}.$$

Hence two main eigenvalues of $A^*(G)$ are

$$\lambda^*_{1,2} = \frac{n_1 + n_2 - 2 - 2r_1 - 2r_2 \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = [n_1 + n_2 - 2 - 2(r_1 + r_2)]^2 - 4[(n_1 - 1 - 2r_1)(n_2 - 1 - 2r_2) - n_1 n_2].$$

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HOUQING ZHOU
DEPARTMENT OF MATHEMATICS
SHAOYANG UNIVERSITY
HUNAN 422000
P.R. CHINA
E-mail address: zhouhq2004@163.com