Mathematica Moravica
Vol. **12-1** (2008), 25–36

Common Fixed Points for Generalized Affine and Subcompatible Mappings with Application

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Abstract. Common fixed point results for generalized affine mapping and a class of $\mathcal{I}$-nonexpansive noncommuting mappings, known as, subcompatible mappings, satisfying $(E.A)$ property have been obtained in the present work. Some useful invariant approximation results have also been determined by its application. These results extend and generalize various existing known results with the aid of more general class of noncommuting mappings, Ciric's contraction type condition and generalized affine mapping in the literature.

1. Introduction

Fixed point theorems have been applied in the field of invariant approximation theory since last four decades and several interesting and valuable results have been studied.

Meinardus [8] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [3] obtained a celebrated result and generalized the Meinardus's result. Later, several results [5, 13, 18] have been proved in the direction of Brosowski [3]. In the year 1988, Sahab, Khan and Sessa [10] extended the result of Hicks and Humpheries [5] and Singh [13] by considering one linear and the other nonexpansive mappings. Al-Thagafi [2] generalized result of Sahab, Khan and Sessa [10] and proved some results on invariant approximations for commuting mappings. The introduction of non-commuting maps to this area, Shahzad [11, 12] further extended Al-Thagafi's results and obtained a number of results regarding invariant approximation.

Recently, Nashine [9] used the concept of affine with respect to a point which was introduced by Vijayaraju and Marudai [19] and results of Jungck and Sessa [7] and many others have been improved and generalized for family of commuting mappings.

2000 *Mathematics Subject Classification.* Primary: 41A50; Secondary: 47H10, 54H25.
Key words and phrases. Best approximation, weakly compatible maps, subcompatible maps, property $(E.A)$, generalized affine map, generalized affine map.

Attempt has been made to find existence results on common fixed point theorem to generalize affine mapping and a class of $\mathcal{I}$-nonexpansive noncommuting maps satisfying new property known as $(E.A)$ property which is further applied to prove some useful invariant approximation result. Another fixed point result with four subcompatible mappings under a contractive condition in terms of the function ψ has also been given. These results extend the results of Nashine [9] in the sense that the general class of $\mathcal{I}$-nonexpansive noncommuting have been used for pair of mappings. These results extend and generalize the results of Shahzad [11, 12] with the aid of more general class of noncommuting, known as, subcompatible mappings, instead of $\mathcal{R}$-subcommuting or $\mathcal{R}$-subweakly commuting mappings and generalized affine mapping instead of linear or affine mapping. By doing this, some known results of Al-Thagafi [2], Brosowski [3], Meinardus [8], Sahab, Khan and Sessa [10] and Singh [13, 14] have also been extended in normed spaces and a approach has been made to give a new direction to the line of investigation initiated [3].

2. Preliminaries

In the material to be produced here, the following definitions have been used:

Definition 2.1 ([15]). Let $\mathcal{M}$ be a subset of $\mathcal{X}$ be a metric space. Let $x_0 \in \mathcal{X}$. An element $y \in \mathcal{M}$ is called a best approximant to $x_0 \in \mathcal{X}$, if

$$d(x_0, y) = dist(x_0, \mathcal{M}) = \inf\{d(x_0, z) : z \in \mathcal{M}\}.$$

Let $\mathcal{P}_{\mathcal{M}}(x_0)$ be the set of best $\mathcal{M}-$approximants to x_0 and so

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{z \in \mathcal{M} : d(x_0, z) = dist(x_0, \mathcal{M})\}.$$

Definition 2.2 ([15]). Let $\mathcal{X}$ be a metric space. A set $\mathcal{M}$ in $\mathcal{X}$ is said to be convex, if $\lambda x + (1-\lambda)y \in \mathcal{M}$, whenever $x, y \in \mathcal{M}$ and $0 \leq \lambda \leq 1$.

A set $\mathcal{M}$ in $\mathcal{X}$ is said to be starshaped, if there exists at least one point $p \in \mathcal{M}$ such that the line segment $[x, p]$ joining x to p is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$ (that is $\lambda x + (1-\lambda)p \in \mathcal{M}$, for all $x \in \mathcal{M}$ and $0 < \lambda < 1$). In this case p is called the starcenter of $\mathcal{M}$.

Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2.3. [19] Let $\mathcal{M}$ be a convex subset of metric space $\mathcal{X}$. Then self-mapping $\mathcal{T}$ of $\mathcal{M}$ is said to be affine if:

$$\mathcal{T}\big(\lambda x + (1-\lambda)y\big) = \lambda\mathcal{T}(x) + (1-\lambda)\mathcal{T}(y),$$

for all $x, y \in \mathcal{M}$ and $\lambda \in (0, 1)$.

Further, definition providing the notion of affine with respect to a point, which is a generalization of an affine mapping, introduced by Vijayaraju and Marudai [19] may be written as:

Definition 2.4 ([19])**.** Let $\mathcal{M}$ be a nonempty, convex subset of metric space $\mathcal{X}$, and let $p \in \mathcal{M}$. A self-mapping $\mathcal{T}$ of $\mathcal{M}$ is said to be affine with respect to p if:

$$\mathcal{T}(\lambda x + (1-\lambda)p) = \lambda\mathcal{T}(x) + (1-\lambda)\mathcal{T}(p),$$

for all $x \in \mathcal{M}$ and $\lambda \in (0,1)$.

The following example shows that an affine mapping with respect to a point need not be affine.

Example 2.1 ([19])**.** Let $\mathcal{X} = \mathbb{R}$ and let $\mathcal{M} = [0,1]$. Define $\mathcal{T}$ on $\mathcal{M}$ by

$$\mathcal{T}x = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1. \end{cases}$$

Then we have

$$\mathcal{T}(\lambda x + (1-\lambda)1/2) = \begin{cases} 1, & \text{if } x \in [0,1) \text{ and } \lambda \in (0,1), \\ 0, & \text{if } x = 1 \text{ and } \lambda = 1. \end{cases}$$

If $x \in [0,1)$ and $\lambda \in (0,1)$, then $\mathcal{T}(x) = 1 = \mathcal{T}(1/2)$ and therefore

$$\mathcal{T}(\lambda x + (1-\lambda)1/2) = 1 = \lambda\mathcal{T}(x) + (1-\lambda)\mathcal{T}(1/2).$$

If $x = 1$ and $\lambda = 1$, then

$$\mathcal{T}(\lambda x + (1-\lambda)1/2) = 0 = \lambda\mathcal{T}(x) + (1-\lambda)\mathcal{T}(1/2).$$

Therefore $\mathcal{T}$ is affine with respect to $1/2$. If $x = 1$ and $\lambda = 1/2$, then

$$\mathcal{T}(\lambda x + (1-\lambda)1/2) = T(3/4) = 1 \neq 1/2 = \lambda\mathcal{T}(1) + (1-\lambda)\mathcal{T}(1/2).$$

Hence, $\mathcal{T}$ is not affine.

Definition 2.5 ([6])**.** A pair $(\mathcal{T},\mathcal{I})$ of self-mappings of a metric space $\mathcal{X}$ is said to be compatible, if $d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{X}$.

Every commuting pair of mappings is compatible but the converse is not true in general.

Jungck [7] introduced the concept of weakly compatible maps as follows:

Definition 2.6. A pair $(\mathcal{I},\mathcal{T})$ of self-mappings of a metric space $\mathcal{X}$ is said to be weakly compatible, if they commute at there coincidence points, i.e., if $\mathcal{T}u = \mathcal{I}u$ for some $u \in \mathcal{X}$, then $\mathcal{T}\mathcal{I}u = \mathcal{I}\mathcal{T}u$.

It is easy to see that compatible maps are weakly compatible.

Definition 2.7. Suppose that $\mathcal{M}$ is p-starshaped with $p \in \mathcal{F}(\mathcal{I})$ and is both $\mathcal{T}$- and $\mathcal{I}$-invariant. Then $\mathcal{T}$ and $\mathcal{I}$ are called $\mathcal{R}$-subcommuting on $\mathcal{M}$, if for all $x \in \mathcal{M}$ there exists a real number $\mathcal{R} > 0$ such that $d(\mathcal{I}\mathcal{T}x, \mathcal{T}\mathcal{I}x) \leq (\frac{\mathcal{R}}{k})d(((1-k)p + k\mathcal{T}x), \mathcal{I}x)$ for each $k \in (0,1]$. If $\mathcal{R} = 1$, then the maps are called 1-subcommuting. The $\mathcal{I}$ and $\mathcal{T}$ are called $\mathcal{R}$-subweakly commuting on $\mathcal{M}$, if for all $x \in \mathcal{M}$ there exists a real number $\mathcal{R} > 0$ such that $d(\mathcal{I}\mathcal{T}x, \mathcal{T}\mathcal{I}x) \leq \mathcal{R}d(\mathcal{I}x, [p, \mathcal{T}x])$, where $[p,x] = (1-k)p + kx : 0 \leq k \leq 1$.

Definition 2.8. Suppose that $\mathcal{M}$ is p-starshaped with $p \in \mathcal{F}(\mathcal{I})$, define $\bigwedge_p(\mathcal{I},\mathcal{T}) = \{\bigwedge(\mathcal{I},\mathcal{T}) : 0 \leq k \leq 1\}$ where $\mathcal{T}_k x = (1-k)q + k\mathcal{T}x$ and $\bigwedge(\mathcal{I},\mathcal{T}) = \{\{x_n\} \subset \mathcal{M} : \lim_n \mathcal{I}x_n = \lim_n \mathcal{T}_k x_n = t \in \mathcal{M} \to \lim_n d(\mathcal{I}\mathcal{T}_k x_n, \mathcal{T}_k\mathcal{I}x_n) = 0\}$, for all sequences $\{x_n\} \in \bigwedge_p(\mathcal{I},\mathcal{T})$. Then $\mathcal{I}$ and $\mathcal{T}$ are called subcompatible [16, 17] if

$$\lim_n d(\mathcal{I}\mathcal{T}x_n, \mathcal{T}\mathcal{I}x_n) = 0$$

for all sequences $x_n \in \bigwedge_p(\mathcal{I},\mathcal{T})$.

Obviously, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.2. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [1,\infty)$. Let $\mathcal{I}(x) = 2x-1$ and $\mathcal{T}(x) = x^2$, for all $x \in \mathcal{M}$. Let $p = 1$. Then $\mathcal{M}$ is p-starshaped with $\mathcal{I}p = p$. Note that $\mathcal{I}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{M}$ with $\lim_n x_n = 2$, we have, $\lim_n \mathcal{I}x_n = \lim_n \mathcal{T}_{\frac{2}{3}} x_n = 3 \in \mathcal{M} \to \lim_n \|\mathcal{I}\mathcal{T}_{\frac{2}{3}} x_n - \mathcal{T}_{\frac{2}{3}}\mathcal{I}x_n\| = 0$. However, $\lim_n \|\mathcal{I}\mathcal{T}x_n - \mathcal{T}\mathcal{I}x_n\| = 0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are not subcompatible maps.

Note that $\mathcal{R}$-subweakly commuting and $\mathcal{R}$-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.3. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [0,\infty)$. Let $\mathcal{I}(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $\mathcal{I}x = x$ if $x = 1$, and $\mathcal{T}(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $\mathcal{T}x = x^2$ if $x = 1$. Then $\mathcal{M}$ is 1-starshaped with $\mathcal{I}1 = 1$ and $\bigwedge_p(\mathcal{I},\mathcal{T}) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that $\mathcal{I}$ and $\mathcal{T}$ are subcompatible but not $\mathcal{R}$-weakly commuting for all $\mathcal{R} > 0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are neither $\mathcal{R}$-subweakly commuting nor $\mathcal{R}$-subcommuting maps.

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, $\mathcal{R}$-weak commutativity and other variants of commutativity of maps as well.

Example 2.4. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [0,\infty)$. Let $\mathcal{I}(x) = 1 + x$ and $\mathcal{T}(x) = 2 + x^2$. Then $|\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x| = 2x$ and $|\mathcal{I}x - \mathcal{T}x| = |x^2 - x + 1|$. Thus the pair $(\mathcal{I},\mathcal{T})$ is not weakly commuting on $\mathcal{M}$ with respect to usual metric. But if $\mathcal{X}$ is endowed with the discrete metric d, then $d(\mathcal{I}\mathcal{T}x, \mathcal{T}\mathcal{I}x) = 1 = d(\mathcal{I}x, \mathcal{T}x)$ for $x > 1$. Thus the pair $(\mathcal{I},\mathcal{T})$ is weakly commuting on $\mathcal{M}$ with respect to discrete metric.

Further, definition providing the notion of $(E.A)$ property introduced by Aamri and El Moutawakil [1] may be written as:

Definition 2.9 ([1])**.** A pair $(\mathcal{I},\mathcal{T})$ of self-mappings of a normed space $\mathcal{X}$ is said to satisfy $(E.A)$ property, if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} \mathcal{I}x_n = t$$

for some $t \in \mathcal{X}$.

Example 2.5. Let $\mathcal{X} = [0, +\infty[$. Define $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = \frac{x}{4} \quad \text{and} \quad \mathcal{I}x = \frac{3x}{4} \quad \text{for all} \quad x \in \mathcal{X}.$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly

$$\lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} \mathcal{I}x_n = 0.$$

Then $\mathcal{T}$ and $\mathcal{I}$ satisfy $(E.A)$.

Example 2.6. Let $\mathcal{X} = [2, +\infty[$. Define $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = x + 1 \quad \text{and} \quad \mathcal{I}x = 2x + 1 \quad forall x \in \mathcal{X}.$$

Suppose that property $(E.A)$ holds; then there exists in $\mathcal{X}$ a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} \mathcal{I}x_n = t \quad \text{for some} \quad t \in \mathcal{X}.$$

Therefore

$$\lim_{n\to\infty} x_n = t - 1 \quad and \quad \lim_{n\to\infty} x_n = \frac{t-1}{2}.$$

Then $t = 1$, which is a contradiction since 1 does not belongs to $\mathcal{X}$. Hence $\mathcal{T}$ and $\mathcal{I}$ do not satisfy $(E.A)$.

Throughout, this paper $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$) denotes the set of fixed points of mapping $\mathcal{T}$ (resp. $\mathcal{I}$).

The following result would also be used in the sequel:

Theorem 2.1 ([1])**.** *Let $\mathcal{T}$ and $\mathcal{I}$ be two weakly compatible self mappings of a metric space $(\mathcal{X}, d)$ such that*

(i) *$\mathcal{T}$ and $\mathcal{I}$ satisfy the property $(E.A)$,*

(ii) $d(\mathcal{T}x, \mathcal{T}y) < \max\Big\{d(\mathcal{I}x, \mathcal{I}y), \frac{1}{2}\big[d(\mathcal{T}x, \mathcal{I}x) + d(\mathcal{T}y, \mathcal{I}y)\big], \frac{1}{2}\big[d(\mathcal{T}y, \mathcal{I}x) + d(\mathcal{T}x, \mathcal{I}y)\big]\Big\}$*, for all $x \neq y \in \mathcal{X}$,*

(iii) $\mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$.

If $\mathcal{I}(\mathcal{X})$ or $\mathcal{T}(\mathcal{X})$ is a complete subspace of $\mathcal{X}$, then $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point.

3. Main Result

First, a more general result in common fixed point theory for more general class of noncommuting and generalized affine mappings is presented below:

Theorem 3.1. *Let $\mathcal{M}$ be a compact subset of normed space $\mathcal{X}$. Suppose $(\mathcal{T}, \mathcal{I})$ continuous and subcompatible self-mappings of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$. Suppose $\mathcal{I}$ is affine with respect to p, $p \in \mathcal{F}(\mathcal{I})$ and $\mathcal{M}$ is* *p-starshaped. If $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property (E.A) for each $0 \leq \lambda \leq 1$; here $\mathcal{T}_\lambda x = \lambda \mathcal{T} x + (1-\lambda)p$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy*

(1)
$$\|\mathcal{T}x - \mathcal{T}y\| \leq \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, \frac{1}{2}\big[dist([\mathcal{T}x,p],\mathcal{I}x) + dist([\mathcal{T}y,p],\mathcal{I}y)\big], \frac{1}{2}\big[dist([\mathcal{T}y,p],\mathcal{I}x) + dist([\mathcal{T}x,p],\mathcal{I}y)\big]\Big\},$$

for all $x \neq y \in \mathcal{M}$, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Proof. Choose a sequence $\{k_n\} \subset [0,1)$ with $k_n \to 1$ as $n \to \infty$. Define for each $n \geq 1$ and for all $x \in \mathcal{M}$, a mapping $\mathcal{T}_n$ by

$$\mathcal{T}_n x = k_n \mathcal{T} x + (1-k_n)p.$$

Then each $\mathcal{T}_n$ is a self-mapping of $\mathcal{M}$ and for each, $\mathcal{T}_n(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$, since $\mathcal{I}$ is affine with respect to p, $p \in \mathcal{F}(\mathcal{I})$ and $\mathcal{T}_n(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$. The subcompatibility of the pair $(\mathcal{I},\mathcal{T})$ implies that

$$\begin{aligned} 0 &\leq \lim_n \|\mathcal{T}_n \mathcal{I} x_m - \mathcal{I}\mathcal{T}_n x_m\| \leq \\ &\leq \lim_m k_n \|\mathcal{T}\mathcal{I}x_m - \mathcal{I}\mathcal{T}x_m\| + \lim_m (1-k_n)\|p - \mathcal{I}p\| = 0, \end{aligned}$$

for any $\{x_m\} \subset \mathcal{M}$ with $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{I} x_m = t \in \mathcal{M}$.

Thus $(\mathcal{T}_n, \mathcal{I})$ are compatible and hence weakly compatible on $\mathcal{M}$ for each n. Also

$$\begin{aligned} \|\mathcal{T}_n x - \mathcal{T}_n y\| &= k_n \|\mathcal{T}x - \mathcal{T}y\| \leq \\ &\leq k_n \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, \frac{1}{2}[dist([\mathcal{T}x,p],\mathcal{I}x) + dist([\mathcal{T}y,p],\mathcal{I}y)], \\ &\quad \frac{1}{2}[dist([\mathcal{T}y,p],\mathcal{I}x) + dist([\mathcal{T}x,p],\mathcal{I}y)]\Big\} \leq \\ &= k_n \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, \frac{1}{2}[\|\mathcal{T}_n x - \mathcal{I}x\| + \|\mathcal{T}_n y - \mathcal{I}y\|], \\ &\quad \frac{1}{2}[\|\mathcal{T}_n y - \mathcal{I}x\| + \|\mathcal{T}_n x - \mathcal{I}y\|]\Big\} \\ &< \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, \frac{1}{2}[\|\mathcal{T}_n x - \mathcal{I}x\| + \|\mathcal{T}_n y - \mathcal{I}y\|], \\ &\quad \frac{1}{2}[\|\mathcal{T}_n y - \mathcal{I}x\| + \|\mathcal{T}_n x - \mathcal{I}y\|]\Big\} \end{aligned}$$

for all $x, y \in \mathcal{M}$. Thus, Theorem 2.1 guarantees that $\mathcal{M} \cap \mathcal{F}(\mathcal{T}_n) \cap \mathcal{F}(\mathcal{I}) = \{x_n\}$ for some $x_n \in \mathcal{M}$.

Also, since $\mathcal{M}$ is compact, there exists a subsequence of $\{x_n\}$ in $\mathcal{M}$, denoted by $\{x_m\}$, converging to a point, say, $y \in \mathcal{M}$ and hence $\mathcal{T}x_m \to \mathcal{T}y$. The continuity of $\mathcal{T}$ gives

$$x_m = \mathcal{T}_m x_m = k_m \mathcal{T} x_m + (1-k_m)p \to \mathcal{T}y$$

and thus the uniqueness of the limit implies $\mathcal{T}y = y$ giving thereby $y \in \mathcal{M} \cap \mathcal{F}(\mathcal{T})$. By the continuity of $\mathcal{I}$, we have

$$\mathcal{I}y = \mathcal{I}\left(\lim_{m\to\infty} x_m\right) = \lim_{m\to\infty} \mathcal{I}x_m = \lim_{m\to\infty} x_m = y,$$

i.e., $\mathcal{I}y = y$. Hence $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$. $\square$

An immediate consequence of the Theorem 3.1 is as follows:

Corollary 3.1.1. *Let $\mathcal{M}$ be a compact subset of normed space $\mathcal{X}$. Suppose $(\mathcal{T},\mathcal{I})$ continuous and subcompatible self-mappings of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$. Suppose $\mathcal{I}$ is affine with respect to p, $p \in \mathcal{F}(\mathcal{I})$ and $\mathcal{M}$ is p-starshaped. If $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property $(E.A)$ for each $0 \leq \lambda \leq 1$; here $\mathcal{T}_\lambda x = \lambda\mathcal{T}x + (1-\lambda)p$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy*

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\| \leq \max\Big\{&\|\mathcal{I}x - \mathcal{I}y\|, \frac{1}{2}dist([\mathcal{T}x,p],\mathcal{I}x), \frac{1}{2}dist([\mathcal{T}y,p],\mathcal{I}y),\\ &\frac{1}{2}dist([\mathcal{T}y,p],\mathcal{I}x), \frac{1}{2}dist([\mathcal{T}x,p],\mathcal{I}y)\Big\},\end{aligned} \tag{2}$$

for all $x \neq y \in \mathcal{M}$, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Next result deals with four subcompatible mappings under a contractive condition in terms of the function ψ. Various conditions on ψ have been studies by different authors.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ satisfy the following conditions:

(i) ψ is nondecreasing on $\mathbb{R}^+$.
(ii) $0 < \psi < t$, for each $t \in (0, +\infty)$.

The following result of Aamri and El Moutawakil [1] would also be needed in the sequel:

Theorem 3.2 ([1])**.** *Let $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{I}$ be self-mappings of a metric space $(\mathcal{X}, d)$ such that*

(i) *$\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{I}$, for all $(x,y) \in \mathcal{X}^2$ satisfy*

$$d(\mathcal{T}x, \mathcal{S}y) \leq \psi(\max\{d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{S}y), d(\mathcal{J}y, \mathcal{S}y)\}), \tag{3}$$

(ii) *$(\mathcal{T},\mathcal{I})$ and $(\mathcal{S},\mathcal{J})$ are weakly compatibles,*
(iii) *$(\mathcal{T},\mathcal{I})$ and $(\mathcal{S},\mathcal{J})$ satisfy the property $(E.A)$,*
(iv) *$\mathcal{T}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$.*

If the range of the one of the mappings $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ or $\mathcal{I}$ is a complete subspace of $\mathcal{X}$, then $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{I}$ have a unique common fixed point.

Theorem 3.3. *Let $\mathcal{M}$ be a compact subset of normed space $\mathcal{X}$. Let $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{I}$ be continuous self-mappings of $\mathcal{M}$ and the pairs $(\mathcal{T},\mathcal{I})$ and $(\mathcal{S},\mathcal{J})$ are subcompatible such that $\mathcal{T}(\mathcal{M}) \subset \mathcal{J}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$. Suppose $\mathcal{I}$ and $\mathcal{J}$ are affine with respect to p, $p \in \mathcal{F}(\mathcal{I})$, and $\mathcal{M}$ is p-starshaped. If the pairs $(\mathcal{T}_\lambda, \mathcal{I})$ and $(\mathcal{S}_\lambda, \mathcal{J})$ satisfy the property $(E.A)$ for each $0 \leq \lambda \leq 1$;*

here $\mathcal{T}_\lambda x = \lambda \mathcal{T}x + (1-\lambda)p$ *and* $\mathcal{S}_\lambda x = \lambda \mathcal{S}x + (1-\lambda)p$ *and* $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ *and* $\mathcal{I}$ *satisfy*

$$(4)\quad \|\mathcal{T}x - \mathcal{S}y\| \le \psi(\max\{\|\mathcal{I}x - \mathcal{J}y\|, dist(\mathcal{I}x, [\mathcal{S}y, p]), dist(\mathcal{J}y, [\mathcal{S}y, p])\})$$

for all $x \neq y \in \mathcal{M}$, *then* $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \emptyset$.

Proof. As in the Theorem 3.1, it can be defined $\mathcal{T}_n$ and $\mathcal{S}_n$, and proof that the pairs $(\mathcal{T}_n, \mathcal{I})$ and $(\mathcal{S}_n, \mathcal{J})$ are weakly compatible and (3). Hence, by Theorem 3.2, there exists $x_n = \mathcal{F}(\mathcal{T}_n) = \mathcal{F}(\mathcal{I}) = \mathcal{F}(\mathcal{S}_n) = \mathcal{F}(\mathcal{J})$ for some $x_n \in \mathcal{M}$. Let $\mathcal{I}$ be compact (same concerns the cases when $\mathcal{T}$ or $\mathcal{S}$ or $\mathcal{J}$ is compact). As $\{x_n\}$ is bounded, so $\{\mathcal{I}x_n\}$ has a subsequence $\{\mathcal{I}x_m\}$ converging to y in $\mathcal{M}$. Now,

$$x_m = \mathcal{I}x_m = \mathcal{T}_m x_m = k_m \mathcal{T}x_m + (1-k_m)p$$

and

$$x_m = \mathcal{J}x_m = \mathcal{S}_m x_m = k_m \mathcal{S}x_m + (1-k_m)p.$$

The continuities of $\mathcal{T}$, $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ implies that $y = \mathcal{T}y = \mathcal{S}y = \mathcal{I}y = \mathcal{J}y$. Hence $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \emptyset$. This completes the proof. □

As an application of Theorem 3.1, the following is more general result in invariant approximation theory with the aid of subcompatible and generalized affine mappings:

Theorem 3.4. *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$. *Let* $\mathcal{M}$ *be subset of* $\mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ *and* $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. *Suppose* $\mathcal{I}$ *is affine with respect to* p *on* $\mathcal{P}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{P}_{\mathcal{M}}(x_0)$ *is compact and p-starshaped,* $\mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$, $\mathcal{T}_\lambda$ *and* $\mathcal{I}$ *satisfy the property (E.A) for each* $0 \le \lambda \le 1$; *here* $\mathcal{T}_\lambda x = \lambda \mathcal{T}x + (1-\lambda)p$. *If the pair* $(\mathcal{T}, \mathcal{I})$ *is continuous, subcompatible and satisfy for all* $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$

$$(5)\qquad \|\mathcal{T}x - \mathcal{T}y\| \le \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\| & \text{if} \quad y = x_0, \\ \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, & \\ \frac{1}{2}\big[dist([\mathcal{T}x, p], \mathcal{I}x) + & \\ dist([\mathcal{T}y, p], \mathcal{I}y)\big], & \\ \frac{1}{2}[dist([\mathcal{T}y, p], \mathcal{I}x) + & \\ dist([\mathcal{T}x, p], \mathcal{I}y)]\Big\}, & \text{if} \quad y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases}$$

then $\mathcal{P}_{\mathcal{M}}(x_0) \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then $y \in \partial\mathcal{M}$ and so $\mathcal{T}y \in \mathcal{M}$, because $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$, we have

$$\|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \le \|\mathcal{I}y - \mathcal{I}x_0\| = \|\mathcal{I}y - x_0\| = dist(x_0, \mathcal{M}).$$

This shows that $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0) = \mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0))$. Now Theorem 3.1 guarantees that

$$\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset.$$

This completes the proof. □

An immediate consequence is as follows:

Corollary 3.4.1. *Let $\mathcal{X}$ be a normed space and $\mathcal{T},\mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine with respect to p on $\mathcal{P}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{P}_{\mathcal{M}}(x_0)$ is compact, p-starshaped, and $\mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$ and $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy satisfy the property $(E.A)$ for each $0 \le \lambda \le 1$; here $\mathcal{T}_\lambda x = \lambda\mathcal{T}x + (1-\lambda)p$. If the pair $\{\mathcal{T},\mathcal{I}\}$ is continuous, subcompatible and satisfy for all $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|, & \text{if} \quad y = x_0, \\ \max\Big\{ \|\mathcal{I}x - \mathcal{I}y\|, \\ \frac{1}{2} dist([\mathcal{T}x, p], \mathcal{I}x), \\ \frac{1}{2} dist([\mathcal{T}y, p], \mathcal{I}y), \\ \frac{1}{2} dist([\mathcal{T}y, p], \mathcal{I}x), \\ \frac{1}{2} dist([\mathcal{T}x, p], \mathcal{I}y)\Big\}, & \text{if} \quad y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases} \tag{6}$$

then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Define $\mathcal{C}_{\mathcal{M}}^{\mathcal{I}}(x_0) = \{x \in \mathcal{M} : \mathcal{I}x \in \mathcal{P}_{\mathcal{M}}(x_0)\}$ and $\mathcal{D}^{\mathcal{I}_{\mathcal{M}}}(x_0) = \mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}^{\mathcal{I}_{\mathcal{M}}}(x_0)$ [2].

Theorem 3.5. *Let $\mathcal{X}$ be a normed space and $\mathcal{T},\mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine with respect to p on $\mathcal{D}^* = \mathcal{D}_{\mathcal{M}}^{\mathcal{I}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$ and $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property $(E.A)$ for each $0 \le \lambda \le 1$; here $\mathcal{T}_\lambda x = \lambda\mathcal{T}x + (1-\lambda)p$. If the pair $(\mathcal{T},\mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x \in \mathcal{D}^* \cup \{x_0\}$*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|, & \text{if} \quad y = x_0, \\ \max\Big\{ \|\mathcal{I}x - \mathcal{I}y\|, \\ \frac{1}{2}[dist([\mathcal{T}x, p], \mathcal{I}x) + \\ dist([\mathcal{T}y, p], \mathcal{I}y)], \\ \frac{1}{2}[dist([\mathcal{T}y, p], \mathcal{I}x) + \\ dist([\mathcal{T}x, p], \mathcal{I}y)]\Big\}, & \text{if} \quad y \in \mathcal{D}^*, \end{cases} \tag{7}$$

then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Proof. First, we show that $\mathcal{T}$ is a self map on $\mathcal{D}^*$, i.e., $\mathcal{T} : \mathcal{D}^* \to \mathcal{D}^*$. Let $y \in \mathcal{D}^*$, then $\mathcal{I}y \in \mathcal{D}^*$, since $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$. By the definition of $\mathcal{D}^*$, $y \in \partial\mathcal{M}$. Also $\mathcal{T}y \in \mathcal{M}$, since $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$,

$$\|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \le \|\mathcal{I}y - \mathcal{I}x_0\|.$$

As $\mathcal{I}x_0 = x_0$,

$$\|\mathcal{T}y - \mathcal{T}x_0\| \leq \|\mathcal{I}y - x_0\| = dist(x_0, \mathcal{M}),$$

since $\mathcal{I}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y$ is also closest to x_0, so $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$,

$$\begin{aligned}\|\mathcal{I}\mathcal{T}y - x_0\| = \|\mathcal{I}\mathcal{T}y - \mathcal{I}x_0\| &\leq \|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \\ &\leq \|\mathcal{I}y - \mathcal{I}x_0\| = \|\mathcal{I}y - x_0\|.\end{aligned}$$

Thus, $\mathcal{I}\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y \in \mathcal{C}^{\mathcal{I}_{\mathcal{M}}}(x_0)$ and hence $\mathcal{T}y \in \mathcal{D}^*$. So $\mathcal{T}$ and $\mathcal{I}$ are selfmaps on $\mathcal{D}^*$. Hence, all the condition of the Theorem 3.1 are satisfied. Thus, there exists $z \in \mathcal{P}_{\mathcal{M}}(x_0)$ such that $z = \mathcal{I}z = \mathcal{T}z$. □

Next, a consequence of the Theorem 3.5 is as follows:

Corollary 3.5.1. *Let $\mathcal{X}$ be a normed space and $\mathcal{T},\mathcal{I} : \mathcal{X} \to X$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine with respect to p on $\mathcal{D}^* = \mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, and $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property $(E.A)$ for each $0 \leq \lambda \leq 1$; here $\mathcal{T}_\lambda x = \lambda\mathcal{T}x + (1-\lambda)p$. If the pair $(\mathcal{T},\mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x \in \mathcal{D}^* \cup \{x_0\}$*

$$(8) \quad \|\mathcal{T}x - \mathcal{T}y\| \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|, & if \quad y = x_0, \\ \max\Big\{ \|\mathcal{I}x - \mathcal{I}y\|, & \\ \frac{1}{2}dist([\mathcal{T}x,p],\mathcal{I}x), \frac{1}{2}dist([\mathcal{T}y,p],\mathcal{I}y), & \\ \frac{1}{2}dist([\mathcal{T}y,p],\mathcal{I}x), \frac{1}{2}dist([\mathcal{T}x,p],\mathcal{I}y)\Big\}, & if \quad y \in \mathcal{D}^*, \end{cases}$$

then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Theorem 3.6. *Let $\mathcal{X}$ be a normed space and $\mathcal{T},\mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine with respect to p on $\mathcal{D}^* = \mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, and $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property $(E.A)$ for each $0 \leq \lambda \leq 1$; here $\mathcal{T}_\lambda x = \lambda\mathcal{T}x + (1-\lambda)p$. If the pair $(\mathcal{T},\mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x \in \mathcal{D}^* \cup \{x_0\}$, (7), then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.*

Proof. Let $x \in \mathcal{D}^*$. Then, $x \in \mathcal{P}_{\mathcal{M}}(x_0)$ and hence $\|x - x_0\| = dist(x_0, \mathcal{M})$. Note that for any $k \in (0,1)$,

$$\|kx_0 + (1-k)x - x_0\| = (1-k)\|x - x_0\| < dist(x_0, \mathcal{M}).$$

It follows that the line segment $\{kx_0 + (1-k)x : 0 < k < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus x is not in the interior of $\mathcal{M}$ and so $x \in \partial\mathcal{M} \cap \mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Along with the lines of the proof of Theorem 3.5, we have the result. □

Corollary 3.6.1. *Let $\mathcal{X}$ be a normed space and $\mathcal{T},\mathcal{I}:\mathcal{X}\to\mathcal{X}$. Let $\mathcal{M}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}\cap\mathcal{M})\subseteq\mathcal{M}$ and $x_0\in\mathcal{F}(\mathcal{T})\cap\mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine with respect to p on $\mathcal{D}^*=\mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(x_0)$, $p\in\mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*)=\mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0)\cup\{x_0\}$, and $\mathcal{T}_\lambda$ and $\mathcal{I}$ satisfy the property $(E.A)$ for each $0\leq\lambda\leq 1$; here $\mathcal{T}_\lambda x=\lambda\mathcal{T}x+(1-\lambda)p$. If the pair $(\mathcal{T},\mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x\in\mathcal{D}^*\cup\{x_0\}$, (8), then $\mathcal{P}_{\mathcal{M}}(x_0)\cap\mathcal{F}(\mathcal{T})\cap\mathcal{F}(\mathcal{I})\neq\emptyset$.*

Remark 3.1. It is observed that $\mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0))\subset\mathcal{P}_{\mathcal{M}}(x_0)$ implies $\mathcal{P}_{\mathcal{M}}(x_0)\subset\mathcal{D}^*$ and hence $\mathcal{D}^*=\mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, Theorem 3.5, Corollary 3.5.1, Theorem 3.6 and Corollary 3.6.1 remain valid when $\mathcal{D}^*=\mathcal{P}_{\mathcal{M}}(x_0)$.

Remark 3.2. Theorem 3.1, Corollary 3.1.1 and Theorem 3.3 contain [2, Theorem 2.2], [4] and [1].

Remark 3.3. In the light of the comment given by Vijayaraju and Marudai [19] that an affine mapping with respect to a point need not be affine, Theorem 3.4 to Corollary 3.6.1 generalize Theorem 3.2 of Al-Thagafi [2], Theorem 3 of Sahab, Khan and Sessa [10] and Singh [13, 14] in the sense that the more generalized noncommuting mappings (subcompatible mappings), generalized relatively nonexpansive maps and generalized affine mapping have been used in place of linearity and relatively nonexpansive commuting maps.

Remark 3.4. With the Remark 3.3 and Example 2.3, our results generalized the results of Shahzad [11, 12].

Remark 3.5. Our results also generalized the results of Nashine [9] in the sense that noncommuting and generalized relatively mappings have been used for pair mappings instead of relative commutative mappings.

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