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Note on Sequence of Exponents of SO-Regular Variability

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Abstract. In this paper we develop the concept of the sequence of exponents of SO-regular variability [9], as a generalization of the sequence of convergence exponents [1].

1. Introduction

Let (a_n) be a nondecreasing sequence of positive numbers. If $a(t) = \sum_{n=1}^{+\infty} a_n t^n$, then it is well known (see [5]) that the asymptotic property

$$\overline{\lim_{t\to 1-}} \frac{a(t)}{a(t^2)} < +\infty \tag{1}$$

is equivalent with property

$$\overline{\lim_{n\to +\infty}} \frac{a_{[\lambda n]}}{a_n} = k_a(\lambda) < +\infty, \quad \lambda > 0. \tag{2}$$

Asymptotic condition (2), in the set of sequences of positive numbers, defines the class of O-regularly varying sequences, i.e. the sequential class OR (see [4]). Karamata's theory of O-regular variability is an essential part of analysis of divergence (see [2]).

An O-regularly varying sequence (a_n) is called SO-regularly varying (see [9, 5]) if there is a $\beta \geq 1$ such that $k_a(\lambda) \leq \beta$, for all $\lambda > 0$. The class of SO-regularly varying sequences is denoted by with SO, and the class of convergence sequences of positive numbers with non zero limit is denoted by K. For classes K, SO and OR the next relations hold:

$$K \subseteq SO \subseteq OR, \quad K \neq SO, \quad SO \neq OR. \tag{3}$$

In [7] Pólya and Szegö considered the concept of sequence of exponents of convergence.

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Definition 1. If (c_n) is a sequence of positive numbers converging to zero, then a sequence of positive numbers (λ_n) is a sequence of exponents of convergence for the sequence (a_n) if for every $\varepsilon > 0$ the series $\sum_{n=1}^{+\infty} a_n^{\lambda_n(1+\varepsilon)}$ converges, and the series $\sum_{n=1}^{+\infty} a_n^{\lambda_n(1-\varepsilon)}$ diverges.

The concept defined above partialy appeared in the papers [3] and [8]. Basic properties of this notion are given in [1] and [10]. Using the idea of definition 1 we now define notion of the sequence of exponents of SO-regular varying convergence.

Definition 2. If (c_n) is a sequence of positive numbers, then a sequence of real numbers (λ_n) is sequence of exponents of SO-regular variability, if for all $\varepsilon \geq 0$, sequence (s_n^1), $s_n^1 = \sum_{k=1}^{n} c_k^{\lambda_k(1+\varepsilon)}$, $n \in \mathbb{N}$, belong to the class SO, and for all $\mu < 0$, sequence (s_n^2), $s_n^2 = \sum_{k=1}^{n} c_n^{\lambda_n(1+\mu)}$ for $n \in \mathbb{N}$, is not in the class SO.

It is clear that for every sequence of positive numbers (a_n) ($a_n \neq 1$, $n \geq n_0$, $n_0 \in \mathbb{N}$) there exists some sequence of exponents of SO-regularly variability. Also, many properties of sequences of exponents of SO-regular variability can be derived from the corresponding properties of sequences of exponents of convergence (see [1]).

Definition 3. Sequence (a_n) is potentialy O-regular varying (then we say that (a_n) belongs to the class PO), if there exists a real number ρ and a sequence $(s_n) \in SO$ such that $a_n = n^\rho \cdot s_n$, $n \in \mathbb{N}$. PO_ρ is the set of all sequences which belong to the class PO for some fixed number ρ.

2. RESULTS

Lemma 1. *Let (a_n) be a sequence of positive numbers and let $b_n = a_n^{\lambda_n}$, $n \in \mathbb{N}$, belongs to the class PO_{-1}. Then (λ_n) is a sequence of exponents of SO-regular variability for the sequence (a_n).*

Proof. The sequence (d_n), $d_n = nb_n$, $n \in \mathbb{N}$, belongs to the class SO. For any $\delta > 1$ $\overline{\lim}_{n\to+\infty} \sup_{\lambda\in[1,\delta]} \dfrac{d_{[\lambda n]}}{d_n} = M(\delta) < +\infty$, $M(\delta) \geq 1$ holds. Let $f(x) = d_{[x]}, x \geq 1$, be function generated by sequence (d_n). It is clear that relation

$$\overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} \leq k_d(\lambda) \cdot M(\delta),$$

for any $\lambda > 0$, $\delta > 1$, holds. Let be $M = \lim_{\delta\to1-} M(\delta)$. So, there exists $\gamma = \beta \cdot M \geq 1$ such that $\overline{\lim}_{x\to+\infty} \dfrac{f(\lambda x)}{f(x)} \leq \gamma$ for all $\lambda > 0$. Also, $f(x) = l(x) \cdot B_0(x)$, $x \geq 1$, where $l(x)$ is a slow varying function, and there exists $A > 0$ such that $\dfrac{1}{A} \leq B_0(x) \leq A$, $x \geq 1$. If $g(t) = b_{[t]}$, $t \geq 1$, then we have

$g(t) = [t]^{-1}l(t)B_0(t)$, $t \geq 1$, and also, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} b_k = B_1(n) \cdot \int_1^{n+1} [t]^{-1} \cdot l(t)\mathrm{d}t = B_1(n) \cdot l_1(n) \quad (n \in N),$$

where $l_1(t)$, $t \geq 1$, is a slow varying function and $\frac{1}{A} \leq B_1(t) \leq A$, $t \geq 1$. So the sequence $(\sum_{k=1}^{n} b_k)$ is an element of the class SO.

If $\mu < 0$, then for $n \geq 1$ we have

$$\sum_{k=1}^{n} b_n^{1+\mu} = B_2(n) \int_1^{n+1} [t]^{-1} \frac{l^{1+\mu}(t)}{[t]^{\mu}} \mathrm{d}t = B_2(n) l_2(n) n^{-\mu},$$

where $l_2(t)$, $t \geq 1$, is slow varying function and

$$\min\{A^{1+\mu}, A^{-1-\mu}\} \leq B_2(t) \leq \max\{A^{1+\mu}, A^{-1-\mu}\}, \quad t \geq 1.$$

So, sequence (s_n^2) (def. 2) is not element of class SO.

If $\varepsilon > 0$, then for $n \geq 1$ we have

$$\sum_{k=1}^{n} b_k^{1+\varepsilon} = B_3(n) \cdot \int_1^{n+1} [t]^{-1-\frac{\varepsilon}{2}} \frac{l^{1+\varepsilon}(t)}{[t]^{\varepsilon/2}} \mathrm{d}t = B_3(n) \cdot p_1(n),$$

where $p_1(t)$, $t \geq 1$, is a function which converges to a positive number for $t \to +\infty$, and $\frac{1}{A^{1+\varepsilon}} \leq B_3(t) \leq A^{1+\varepsilon}$, $t \geq 1$. So, sequence (s_n^1) belongs to the class SO. □

Theorem 1. *Let (a_n) be a sequence of positive numbers and let $b_n = a_n^{\lambda_n}$, $n \in \mathbb{N}$, belong to the class PO. The sequence (λ_n) is a sequence of exponents of SO-regular variability of sequence (a_n) if and only if (b_n) belongs to the class PO_{-1}, i.e.,*

$$b_n = n^{-1} \exp\left\{ \alpha_n + \sum_{k=1}^{n} \frac{\delta_k}{k} \right\}, \quad n \geq 1,$$

where sequence (α_n) is bounded, and sequence (δ_n) converging to zero.

Proof. If (b_n) is an element of the class PO_{-1}, then by Lemma 1, the sequence (λ_n) is a sequence of exponents of SO-regular variability for sequence (a_n). If sequence (b_n) is an element of class PO_ρ, $\rho > -1$, then $b_n = n^\rho \cdot s_n$, $n \in \mathbb{N}$, where (s_n) belongs to class SO. In this case for $n \geq 1$ we have

$$\sum_{k=1}^{n} b_n = B_4(n) \int_1^{n+1} [t]^\rho l(t)\mathrm{d}t = B_4(n) \cdot n^{\rho+1} l_3(n),$$

where $l_3(t)$, $t \geq 1$, is a slow varying function, and $\frac{1}{A} \leq B_4(t) \leq A$, $t \geq 1$. So, the sequence (λ_n) is not a sequence of exponents of SO-regular variability for the sequence (a_n), because the sequence $(\sum_{k=1}^{n} b_k)$ is not an element of the class SO.

If (b_n) belongs to the class PO_ρ, $\rho < -1$, then $b_n = n^\rho S_n$, $n \in \mathbb{N}$, where (s_n) is an element of class SO. Then, $p = -1 - \rho > 0$ and for $\mu = \dfrac{p}{2\rho} < 0$ we have

$$\sum_{k=1}^{n} b_k^{1+\mu} = B_5(n) \int_1^{n+1} [t]^{\rho(1+\mu)} l^{1+\mu}(t)\mathrm{d}t = B_5(n) \cdot \int\limits_1^{n+1} [t]^{\rho+\frac{p}{2}} l^{1+\mu}(t)\mathrm{d}t =$$

$$= B_5(n) \cdot \int_1^{n+1} [t]^{\rho+\frac{3p}{4}} \frac{l^{1+\mu}(t)}{[t]^{\frac{p}{4}}}\mathrm{d}t = B_5(n) \cdot p_2(n),$$

where $p_2(t)$, $t \geq 1$, is a function which converges to a positive limit as t converges to $+\infty$, and $\dfrac{1}{A^{1+\mu}} \leq B_5(t) \leq A^{1+\mu}$, $t \geq 1$. So, sequence (λ_n) is not a sequence of exponents of SO-regular variability for (a_n), because for $\mu = \dfrac{p}{2\rho} < 0$, by the above assumptions, the sequence (s_n^2) belongs to the class SO. If (b_n) is an element of class PO_{-1}, then $b_n = n^{-1} \cdot l(n) \cdot B_0(n)$, $n \geq 1$, where l and B_0 are functions from the proof of Lemma 1. So, for $n \geq 1$, $b_n = n^{-1} \cdot \exp\left\{\alpha_n + \sum_{k=1}^{n} \dfrac{\delta_k}{k}\right\}$ where the sequence (α_n) is bounded and the sequence (δ_n) converges to zero. □

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