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On Super-Associative Algebras with $(2m, m)-$Quasigroup Operations for $m \geq 2$

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Abstract. In this paper super-associative algebras with $(2m, m)-$quasigroup operations are considered. Case $m = 1$ is described in [1]. Super-associative algebras with $n-$quasigroup operations for $n = 3$ Yu. Movsisyan was described in 1984 (cf. [6]). Case $n \geq 3$ was described in [10]. See, also [11].

1. Introduction

Definition 1.1 ([2])**.** Let $n \geq m+1$ $(n, m \in N)$ and $(Q; A)$ be an $(n, m)-$groupoid $(A : Q^n \to Q^m)$. We say that $(Q; A)$ is an $(n, m)-$**group** iff the following statements hold:

(|) For every $i, j \in \{1, \dots, n-m+1\}$, $i < j$, the following law holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$

/: $\langle i, j\rangle-$associative law/[1]; and

(||) For every $i \in \{1, \dots, n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n. \quad ^2$$

is an $(n, m)-$quasigroup. In [4]: weak $(n, m)-$quasigroup.

Remark 1.1. For $m = 1$ $(Q; A)$ is an $n-$group [5]. Cf. Def. 1.1–I in [11].

Remark 1.2. Let $x_1, \dots, x_{2n-m}$ be **subject symbols**, $(n, m \in N,\ n \geq 2m)$ and $X_1, X_2, \dots, X_{2i-1}, X_{2i}$, $i \in \{2, \dots, n-m+1\}$, be $(n, m)-$**ary operational symbols**. Then, we say that

$$X_1(X_2(x_1^n), x_{n+1}^{2n-m}) = X_{2i-1}(x_1^{i-1}, X_{2i}(x_i^{i+n-1}), x_{i+n}^{2n-m}) \tag{1}$$

is a **general** $\langle 1, i\rangle-$**associative law**. (Some of operational simbols in (1) can be equal.)

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[1] $(Q; A)$ is an $(n, m)-$semigroup.
[2] $(Q; A)$

Definition 1.2. Let $(Q; \Sigma)$ be an algebra in which the following holds: $(Q; Z)$ is an $(2m, m)-$quasigroup for every $Z \in \Sigma$. Also let $|\Sigma| \geq 2$. Further on, let $x_1, \ldots, x_{3m}$ be subject symbols, let $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \ldots, m+1\}$, is $|\{X_1, X_2, X_{2i-1}, X_{2i}\}| \geq 2$. Then, we say that $(Q; \Sigma)$ is a **super-associative algebra with** $(2m, m)-$**quasigroup operations** (briefly: $SAA(2m, m)Q$) iff for every substitution of the subject symbols $x_1, \ldots, x_{3m}$ in (1) [3] by elements $\overline{x}_1, \ldots, \overline{x}_{3m}$ of Q and for every substitution of the operational symbols X_1, X_2, X_{2i-1}, X_{2i}, $i \in \{2, \ldots, m+1\}$, in (1) by elements $\overline{X}_1$, $\overline{X}_2$, $\overline{X}_{2i-1}$, $\overline{X}_{2i}$, $i \in \{2, \ldots, m+1\}$, of Σ for all $i \in \{2, \ldots, m+1\}$ the following equality holds:

$$(\overline{1}) \qquad \overline{X}_1(\overline{X}_2(\overline{x}_1^{2m}), \overline{x}_{2m+1}^{3m}) = \overline{X}_{2i-1}(\overline{x}_1^{i-1}, \overline{X}_{2i}(\overline{x}_i^{i+2m-1}), \overline{x}_{i+2m}^{3m}).$$

A immediate consequence of Definition 1.1 and Definition 1.2, is the following proposition:

Proposition 1.1. *If $(Q; \Sigma)$ is a $SAA(2m, m)Q$, then $(Q; Z)$ is an $(2m, m)-$group for every $Z \in \sum$.*
Case $m = 1$ is described in [1].

Proposition 1.2. *Let $(Q; \Sigma)$ be an $SAA(2m, m)Q$. Then the following statements hold:*

$^\circ 1$ $X_1 \neq X_2 \rightarrow \{X_{2i-1}, X_{2i}\} = \{X_1, X_2\}$ *and*
$^\circ 2$ $X_1 = X_2 \rightarrow X_{2i-1} = X_{2i}$
for all $i \in \{2, \ldots, m+1\}$, where $X_1, X_2, X_{2i-1}, X_{2i}$ from 1.1-(1) for $n = 2m$.

Proof. See the Proof of Theorem 2.1-XI in [11]. □

Definition 1.3. We will say that a $SAA(2m, m)Q$ has type $XX(XY)$ iff $^\circ 2(^\circ 1)$ of Proposition 1.2.

2. AUXILIARY PART

Definition 2.1 ([8])**.** Let $n \geq 2m$ and let $(Q; A)$ be an $(n, m)-$groupoid. Also, let $\mathbf{e}$ be mapping of the set Q^{n-2m} into the set Q^m. Then: $\mathbf{e}$ is a $\{1, n-m+1\}-$neutral operation of the $(n, m)-$groupoid $(Q; A)$ iff for every $x_1^m \in Q^m$ and for all sequence a_1^{n-2m} over Q the following equalities hold

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m \quad \text{and}$$
$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m.$$

Remark 2.1. For $m = 1$ $\mathbf{e}$ is an $\{1, n\}-$neutral operation of the $n-$groupoid $(Q; A)$ [7]. For $(n, m) = (2, 1)$, $\mathbf{e}(a_1^\circ)$ *[*$= \mathbf{e}(\emptyset)$*]* is a neutral element of the groupoid $(Q; A)$. Cf. Ch. II [11].

Proposition 2.1 ([8])**.** *Let $(Q; A)$ be an $(n, m)-$groupoid and $n \geq 2m$. Then there is* ***at most one*** *$\{1, n-m+1\}-$neutral operation of $(Q; A)$.*

[3] $n = 2m$

Proof. See in [12]. □

Proposition 2.2 ([8]). *Every $(n,m)-$group, $n \geq 2m$, has a $\{1, n-m+1\}-$neutral operation.*

Proof. See in [12]. □

By Proposition 2.1 and by Proposition 2.2, we have:

Theorem 2.1 ([3]). *Let $(Q;A)$ be an $(n,m)-$group and $n = 2m$. Then there is exactly one $e_1^m \in Q^m$ such that for all $x_1^m \in Q^m$ the following equalities hold*

$$(n) \qquad A(x_1^m, e_1^m) = x_1^m \quad \text{and} \quad A(e_1^m, x_1^m) = x_1^m.$$

Remark 2.2. For $m = 1$, e_1^m is a neutral element of the group $(Q;A)$.

Proof. See in [12]. □

Proposition 2.3 ([3]). *Let $(Q;A)$ be a $(2m,m)-$**group** and let $e_1^m \in Q^m$ satisfying (n) (from Theorem 2.1) for all $x_1^m \in Q^m$. Then, for all $i \in \{0,1,\dots,m\}$ and for every $x_1^m \in Q^m$ the following equality holds*

$$A(x_1^i, e_1^m, x_{i+1}^m) = x_1^m.$$

Proof. See in [12]. □

Theorem 2.2 ([3]). *Let $(Q;A)$ be a $(2m,m)-$**group** and let $e_1^m \in Q^m$ satisfying (n) [from Theorem 2.1] for all $x_1^m \in Q^m$. Then: $e_1 = e_2 = \cdots = e_m$.*

Proof. See in [12]. □

Proposition 2.4 ([9]). *Let $(Q;A)$ be an $(n,m)-$group and $n \geq 2m$. Then there are is mappings $\mathbf{e}$ and ${}^{-1}$, respectively, of the sets Q^{n-2m} and Q^{n-m} into the set Q^m such that the following laws hold in the algebra $(Q; A, {}^{-1}, \mathbf{e})$*

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m,$$
$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m,$$
$$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m}),$$
$$A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m}),$$
$$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m \quad \text{and}$$
$$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m.$$

Proof. See in [12]. See, also 3 in [12]. □

Proposition 2.5 ([9]). *Let $n > m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also, let*

(a) *The $\langle 1,2 \rangle-$associative law holds in $(Q;A)$; and*
(b) *For every $a_1^{n-m} \in Q$ and for each $x_1^m, y_1^m \in Q^m$ the following implication holds*

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \rightarrow x_1^m = y_1^m.$$

Then $(Q;A)$ is an $(n,m)-$semigroup.

Proof. See in [12]. □

Proposition 2.6 ([2])**.** *Let $(Q;A)$ be an $(n,m)-$semigroup and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-m)+m} \in Q$ and for every $t \in \{1,\ldots,i(n-m)+1\}$ the following equality holds*

$$\overset{i+j}{A}\big(x_1^{(i+j)(n-m)+m}\big) = \overset{i}{A}\big(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-m)+m-1}), x_{t+j(n-m)+m}^{(i+j)(n-m)+m}\big).$$

Cf. [12].

Proposition 2.7 ([4])**.** *Let $(Q;A)$ be an $(n,m)-$groupoid and $n \geq m+2$. Also, let the following statements hold:*

$(\widehat{|})$ *$(Q;A)$ is an $(n,m)-$semigroup;*

$(\widehat{||})$ *For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n;$$

and

$(\widehat{|||})$ *For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds*

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then $(Q;A)$ is an $(n,m)-$group.

3. Main part

Definition 3.1. Let $(Q;A)$ be a $(2m,m)-$group and $c_1^m \in Q^m$. Then, we will say that the element c_1^m is a central element of the $(2m,m)-$group $(Q;A)$ iff for every $z_1^m \in\in Q$ and for all $i \in \{1,\ldots,m\}$ the following equality holds

$$A(z_1^i, c_1^m, z_{i+1}^m) = A(c_1^m, z_1^m).$$

Theorem 3.1. *Let $(Q;\Sigma)$ be an $SAA(2m,m)Q$ of the type XX. Also, let A be an arbitrary operation from Σ. Then, for all $B \in \Sigma$ there is a central element $\overset{m}{e}_B$ of the $(2m,m)-$group $(Q;A)$ such that for every $x_1^m, y_1^m \in Q$ the following equalities hold*

$$B(x_1^m, y_1^m) = A(x_1^m, A(y_1^m, \overset{m}{e}_B)) \quad \textit{and}$$
$$(\overset{m}{e}_B)^{-1} = \overset{m}{e}_B,$$

where $^{-1}$ is an inverse operation in the $(2m,m)-$group $(Q;A)$.

Proof. Let A and B two operations from Σ. By Proposition 1.1, $(Q;A)$ and $(Q;B)$ are $(2m,m)-$groups. By Propositions 2.2, 2.3 and Theorems 2.1, 2.2, $(Q;A)$ and $(Q;B)$ have neutral elements, denoted respectively, by
$oversetme$ and $\overset{m}{e}_B$. Let also the inverse operation in $(Q;A)$ be denoted $^{-1}$. See section 3 in [Ušan 2005].

The following statements hold:

$\overset{\circ}{1}$ For all $x_1^m, y_1^m \in Q$ and for every $i \in \{1, \ldots, m\}$ the following equalities hold

$$B(x_1^m, y_1^m) = A(x_1^m, A(y_1^m, \overset{m}{e}_B)) \quad \text{and} \tag{1}$$

$$B(x_1^m, y_1^m) = A(x_1^m, A(y_1^i, \overset{m}{e}_B, y_{i+1}^m)); \tag{$\overline{1}$}$$

$\overset{\circ}{2}$ $\overset{m}{e}_B$ is a central element of the $(2m, m)-$group $(Q; A)$; and
$\overset{\circ}{3}$ $(\overset{m}{e}_B)^{-1} = \overset{m}{e}_B$.

The Proof of $\overset{\circ}{1}$. By Definition 1.3 and by Proposition 1.1, for every $x_1^m, y_1^m, z_1^m \in Q$ the following equalities hold

$$B(x_1^m, B(y_1^m, z_1^m)) \overset{1.1}{=} B(B(x_1^m, y_1^m), z_1^m)) = \overset{1.3}{=} A(x_1^m, A(y_1^m, z_1^m)).$$

Hence, by the substitutions $z_1^m = \overset{m}{e}_B$ and $y_1^m, z_1^m = y_1^i, \overset{m}{e}_B, y_{i+1}^m$ and by Proposition 2.3, we conclude that for every $x_1^m, y_1^m \in Q$ the equalities (1) and ($\overline{1}$) hold.

The proof of $\overset{\circ}{2}$. Since $(Q; B)$ is a $(2m, m)-$group, for every $x_1^{3m} \in Q$ the following equality holds

$$B(B(x_1^m, y_1^m), z_1^m) = B(x_1^m, B(y_1^m, z_1^m)),$$

hence, by the statement $\overset{\circ}{1}$, we conclude that for every $x_1^{3m} \in Q$ the following series of implications hold:

$$B(B(x_1^m, y_1^m), z_1^m) = B(x_1^m, B(y_1^m, z_1^m)) \overset{(1),(\overline{1})}{\Longrightarrow}$$
$$A(A(x_1^m, A(y_1^m, \overset{m}{e}_B)), A(z_1^m, \overset{m}{e}_B)) = A(x_1^m, A(A(y_1^m, A(z_1^i, \overset{m}{e}_B, z_{i+1}^m)), \overset{m}{e}_B) \overset{(|)}{\Longrightarrow}$$
$$A(x_1^m, A(A(y_1^m, \overset{m}{e}_B), A(z_1^m, \overset{m}{e}_B))) = A(x_1^m, A(y_1^m, A(A(z_1^i, \overset{m}{e}_B, z_{i+1}^m), \overset{m}{e}_B))) \overset{(|)}{\Longrightarrow}$$
$$A(x_1^m, A(y_1^m, A(\overset{m}{e}_B, A(z_1^m, \overset{m}{e}_B)))) = A(x_1^m, A(y_1^m, A(A(z_1^i, \overset{m}{e}_B, z_{i+1}^m), \overset{m}{e}_B))) \overset{(|)}{\Longrightarrow}$$
$$A(x_1^m, A(y_1^m, A(A(\overset{m}{e}_B, z_1^m), \overset{m}{e}_B))) = A(x_1^m, A(y_1^m, A(A(z_1^i, \overset{m}{e}_B, z_{i+1}^m), \overset{m}{e}_B))) \overset{(||)}{\Longrightarrow}$$
$$A(A(\overset{m}{e}_B, z_1^m), \overset{m}{e}_B) = A(A(z_1^i, \overset{m}{e}_B, z_{i+1}^m), \overset{m}{e}_B) \overset{(||)}{\Longrightarrow}$$
$$A(\overset{m}{e}_B, z_1^m) = A(z_1^i, \overset{m}{e}_B, z_{i+1}^m).$$

Sketch of the Proof of $\overset{\circ}{3}$. Putting $x_1^m = \overset{m}{e}_B$ and $y_1^m = x_1^m$ in (1), we obtain

$$x_1^m = A(\overset{m}{e}_B, A(x_1^m, \overset{m}{e}_B)),$$

hence, by Proposition 2.4, we conclude that for every $x_1^m \in Q^m$ the following implication hold

$$x_1^m = A(\overset{m}{e}_B, A(x_1^m, \overset{m}{e}_B)) \Rightarrow$$
$$A((\overset{m}{e}_B)^{-1}, x_1^m) = A((\overset{m}{e}_B)^{-1}, A(\overset{m}{e}_B, A(x_1^m, \overset{m}{e}_B))) \overset{(|)}{\Longrightarrow}$$
$$A((\overset{m}{e}_B)^{-1}, x_1^m) = A(A((\overset{m}{e}_B)^{-1}, \overset{m}{e}_B), A(x_1^m, \overset{m}{e}_B)) \overset{2.4}{\Longrightarrow}$$
$$A((\overset{m}{e}_B)^{-1}, x_1^m) = A(\overset{m}{e}, A(x_1^m, \overset{m}{e}_B)) \Longrightarrow$$
$$A((\overset{m}{e}_B)^{-1}, x_1^m) = A(x_1^m, \overset{m}{e}_B).$$

Hence, by the substitution $x_1^m = \overset{m}{e}$, we conclude that the following equality holds

$$(\overset{m}{e}_B)^{-1} = \overset{m}{e}_B.$$

Finaly, by $\overset{\circ}{1}$–$\overset{\circ}{3}$ we conclude that Theorem 3.1 holds. □

Theorem 3.2. *Let $(Q; \Sigma)$ be a $(2m, m)-$group, $A \in \Sigma$, $|\Sigma| \geq 2$ and let for all $B \in \Sigma$ there be a central element $c_1^m (2m, m)-$group $(Q; A)$ such that for every $x_1^m, y_1^m \in Q$ the following equalities hold*

(a) $B(x_1^m, y_1^m) = A(c_1^m, A(x_1^m, y_1^m))$ *and*
(b) $(c_1^m)^{-1} = c_1^m$,

where $^{-1}$ is an inverse operation in the $(2m, m)-$group $(Q; A)$. Then (Q, Σ) is a $SAA(2m, m)Q$ of the type XX.

Proof. The following statements hold:

1° If $B \in \Sigma$, then for every $a_1^m, b_1^m \in Q$ there is exactly one $x_1^m \in Q$ and exactly one $y_1^m \in Q$ such that the following equalities hold

$$B(a_1^m, x_1^m) = b_1^m \quad \text{and} \quad B(y_1^m, a_1^m) = b_1^m;$$

2° If $B \in \Sigma$, then the $\langle 1, 2\rangle-$associative law holds in $(Q; B)$;
3° If $B \in \Sigma$, then $(Q; B)$ is a $(2m, m)-$group; and
4° For all $i \in \{2, \ldots, m+1\}$, for every $x_1^{3m} \in Q$ and for every $C, D \in \Sigma$ the following equality holds

$$C(C(x_1^{2m}), x_{2m+1}^{3m}) = D(x_1^{i-1}, D(x_i^{i+2m-1}), x_{i+2m}^{3m}).$$

Sketch of the Proof of 1°.

a) $$B(a_1^m, x_1^m) = b_1^m \overset{(a)}{\Longleftrightarrow} A(c_1^m, A(a_1^m, x_1^m)) = b_1^m$$
$$\overset{(|)}{\Longleftrightarrow} A(A(c_1^m, a_1^m), x_1^m)) = b_1^m.$$

b) $$B(y_1^m, a_1^m) = b_1^m \overset{(a)}{\Longleftrightarrow} A(c_1^m, A(y_1^m, a_1^m)) = b_1^m$$
$$\overset{3.1}{\Longleftrightarrow} A(A(y_1^m, a_1^m), c_1^m) = b_1^m$$
$$\overset{(|)}{\Longleftrightarrow} A(y_1^m, A(a_1^m, c_1^m)) = b_1^m.$$

Sketch of the Proof of 2°.

$\overline{a}$)

$$\begin{aligned}
B(x_1, B(x_2^m, y_1^m, z_1), z_2^m) &\overset{(a)}{=} A(c_1^m, A(x_1, B(x_2^m, y_1^m, z_1), z_2^m)) \overset{(a)}{=} \\
&= A(c_1^m, A(x_1, A(c_1^m, A(x_2^m, y_1^m, z_1)), z_2^m)) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, x_1, c_1^m, x_2^m, y_1^m, z_1, z_2^m)) \overset{2.6}{=} \\
&= \overset{3}{A}(c_1^m, A(x_1, c_1^m, x_2^m), y_1^m, z_1^m)) \overset{3.1}{=} \\
&= \overset{3}{A}(c_1^m, A(c_1^m, x_1^m), y_1^m, z_1^m)) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, c_1^m, x_1^m, y_1^m, z_1^m)) \overset{2.6}{=} \\
&= \overset{3}{A}(A(c_1^m, c_1^m), x_1^m, y_1^m, z_1^m) \overset{(b)}{=} \\
&= \overset{3}{A}(\overset{m}{e}, x_1^m, y_1^m, z_1^m) \overset{2.6}{=} \\
&= \overset{2}{A}(A(\overset{m}{e}, x_1^m), y_1^m, z_1^m) = \\
&= \overset{2}{A}(x_1^m, y_1^m, z_1^m),
\end{aligned}$$

where $\overset{m}{e}$ is a neutral element of the $(2m, m)-$group $(Q; A)$.

$\overline{b}$)

$$\begin{aligned}
B(B(x_1^m, y_1^m), z_1^m) &\overset{(a)}{=} A(c_1^m, A(B(x_1^m, y_1^m), z_1^m)) \overset{(a)}{=} \\
&= A(c_1^m, A(A(c_1^m, A(x_1^m, y_1^m)), z_1^m)) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, c_1^m, x_1^m, y_1^m, z_1^m) \overset{2.6}{=} \\
&= \overset{3}{A}(A(c_1^m, c_1^m), x_1^m, y_1^m, z_1^m) \overset{(b)}{=} \\
&= \overset{3}{A}(\overset{m}{e}, x_1^m, y_1^m, z_1^m) \overset{2.6}{=} \\
&= \overset{2}{A}(A(\overset{m}{e}, x_1^m), y_1^m, z_1^m) = \overset{2}{A}(x_1^m, y_1^m, z_1^m),
\end{aligned}$$

where $\overset{m}{e}$ is a neutral element of the $(2m, m)-$group $(Q; A)$.

$\overline{c}$) By $\overline{a}$) and by $\overline{b}$), we obtain 2°.

The Proof of 3°. By 1°, 2°, Proposition 2.5 and by Proposition 2.7.

Sketch of the Proof of 4°.

$\overline{\overline{a}}$) $C(C(x_1^m, y_1^m), z_1^m) \overset{b)}{=} A(A((x_1^m, y_1^m), z_1^m)$.

$\overline{\overline{b}}$) $D(D(x_1^m, y_1^m), z_1^m) \overset{b)}{=} A(A((x_1^m, y_1^m), z_1^m)$.

$\overline{\overline{c}}$) By $\overline{\overline{a}}$), $\overline{\overline{b}}$) and by 3°, we obtain 4°.

Finally, by 3°, 4° and by Definition 1.3, we have Theorem 3.2. □

Theorem 3.3. *Let $(Q;\sum)$ be a $(2m,m)-$group, $A \in \Sigma$, $|\Sigma| \geq 2$ and let for all $B \in \Sigma$ there be a central element $c_1^m (2m,m)-$group $(Q;A)$ such that for every $x_1^m, y_1^m \in Q$ the following equality holds*

$$(a) \qquad B(x_1^m, y_1^m) = A(c_1^m, A(x_1^m, y_1^m)).$$

Then (Q,Σ) is a $SAA(2m,m)Q$ of the type XY.

Proof. The following statements hold:

$\overline{1}$ If $B \in \Sigma$, then for every $a_1^m, b_1^m \in Q$ there is exactly one $x_1^m \in Q$ and exactly one $y_1^m \in Q$ such that the following equalities hold
$$B(a_1^m, x_1^m) = b_1^m \quad \text{and} \quad B(y_1^m, a_1^m) = b_1^m.$$

$\overline{2}$ If $B \in \Sigma$, then $\langle 1,2\rangle-$associative law holds in $(Q;B)$.

$\overline{3}$ If $B \in \Sigma$, then $(Q;B)$ is a $(2m,m)-$group.

$\overline{4}$ For all $i \in \{2,\ldots,m+1\}$, for every $x_1^{3m} \in Q$ and for every $C,D \in \Sigma$ the following equality holds
$$B(C(x_1^{2m}), x_{2m+1}^{3m}) = B(x_1^{i-1}, C(x_i^{i+2m-1}), x_{i+2m}^{3m}).$$

Sketch of the Proof of $\overline{1}$. The proof of Theorem 3.2

Sketch of the Proof of $\overline{2}$.

α)

$$\begin{aligned}
B(x_1, B(x_2^m, y_1^m, z_1), z_2^m) &\overset{(a)}{=} A(c_1^m, A(x_1, B(x_2^m, y_1^m, z_1), z_2^m)) \overset{(a)}{=} \\
&= A(c_1^m, A(x_1, A(c_1^m, A(x_2^m, y_1^m, z_1)), z_2^m)) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, x_1, c_1^m, x_2^m, y_1^m, z_1, z_2^m)) \overset{2.6}{=} \\
&= \overset{3}{A}(c_1^m, A(x_1, c_1^m, x_2^m), y_1^m, z_1^m)) \overset{3.1}{=} \\
&= \overset{3}{A}(c_1^m, A(c_1^m, x_1^m), y_1^m, z_1^m) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, c_1^m, x_1^m, y_1^m, z_1^m).
\end{aligned}$$

β)

$$\begin{aligned}
B(B(x_1^m, y_1^m), z_1^m) &\overset{(a)}{=} A(c_1^m, A(B(x_1^m, y_1^m), z_1^m)) \overset{(a)}{=} \\
&= A(c_1^m, A(A(c_1^m, A(x_1^m, y_1^m)), z_1^m)) \overset{2.6}{=} \\
&= \overset{4}{A}(c_1^m, c_1^m, x_1^m, y_1^m, z_1^m)
\end{aligned}$$

γ) By α) and by β), we obtain $\overline{2}$.

The Proof of $\overline{3}$. By $\overline{1}$, $\overline{2}$ Proposition 2.5 and by Proposition 2.7.

Sketch of the Proof of $\overline{4}$.

$\overline{\alpha}$)

$$\begin{aligned} B(C(x_1^m, y_1^m), z_1^m) &\overset{(a)}{=} A(c_1^m, A(C(x_1^m, y_1^m), z_1^m)) \overset{(a)}{=} \\ &= A(c_1^m, A(A(\overline{c}_1^m, A(x_1^m, y_1^m)), z_1^m) \overset{2.6}{=} \\ &= \overset{4}{A}(c_1^m, \overline{c}_1^m, x_1^m, y_1^m, z_1^m). \end{aligned}$$

$\overline{\beta}$)

$$\begin{aligned} B(x_1^{i-1}, C(x_i^m, y_1^m, z_1^{i-1}), z_i^m) &\overset{(a)}{=} A(c_1^m, A(x_1^{i-1}, C(x_i^m, y_1^m, z_1^{i-1}), z_i^m)) \overset{(a)}{=} \\ &= A(c_1^m, A(x_1^{i-1}, A(\overline{c}_1^m, A(x_i^m, y_1^m, z_1^{i-1})), z_i^m)) \overset{2.6}{=} \\ &= \overset{4}{A}(c_1^m, x_1^{i-1}, \overline{c}_1^m, x_i^m, y_1^m, z_1^{i-1}, z_i^m)\{2.6= \\ &= \overset{3}{A}(c_1^m, A(x_1^{i-1}, \overline{c}_1^m, x_i^m), y_1^m, z_i^m) \overset{3.1}{=} \\ &= \overset{3}{A}(c_1^m, A(\overline{c}_1^m, x_1^m), y_1^m, z_i^m) \overset{2.6}{=} \\ &= \overset{4}{A}(c_1^m, \overline{c}_1^m, x_1^m, y_1^m, z_1^m). \end{aligned}$$

$\overline{\gamma}$)

$$\begin{aligned} C(x_1^{i-1}, B(x_i^m, y_1^m, z_1^{i-1}), z_i^m) &\overset{(a)}{=} A(\overline{c}_1^m, A(x_1^{i-1}, B(x_i^m, y_1^m, z_1^{i-1}), z_i^m)) \overset{(a)}{=} \\ &= A(\overline{c}_1^m, A(x_1^{i-1}, A(c_1^m, A(x_i^m, y_1^m, z_1^{i-1})), z_i^m)) \overset{2.6}{=} \\ &= \overset{4}{A}(\overline{c}_1^m, x_1^{i-1}, c_1^m, x_i^m, y_1^m, z_1^{i-1}, z_i^m) \overset{2.6}{=} \\ &= \overset{3}{A}(\overline{c}_1^m, A(x_1^{i-1}, c_1^m, x_i^m), y_1^m, z_1^m) \overset{3.1}{=} \\ &= \overset{3}{A}(\overline{c}_1^m, A(c_1^m, x_1^m), y_1^m, z_1^m) \overset{2.6}{=} \\ &= \overset{4}{A}(\overline{c}_1^m, c_1^m, x_1^m, y_1^m, z_1^m) \overset{2.6}{=} \\ &= \overset{3}{A}(A(\overline{c}_1^m, c_1^m), x_1^m, y_1^m, z_1^m) \overset{3.1}{=} \\ &= \overset{3}{A}(A(c_1^m, \overline{c}_1^m), x_1^m, y_1^m, z_1^m) \overset{2.6}{=} \\ &= \overset{4}{A}(c_1^m, \overline{c}_1^m, x_1^m, y_1^m, z_1^m). \end{aligned}$$

Finally, by $\overline{3}$, $\overline{4}$ and by Definition 1.3, we have Theorem 3.3. □

Remark 3.1. In this paper, $SAA(2m, m)Q$ of the type XY only in one direction are described.

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