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A Common Fixed Point Theorem for Multivalued Mappings Through T-weak Commutativity

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Abstract. In this paper, we prove a common fixed point theorem for a single-valued and the multivalued mappings by using T-weak commutativity condition. We also show that continuity of any mapping is not needed for the existence of the common fixed point.

1. Introduction

In 1976, Jungck [5] proved a common fixed point theorem for commuting maps, generalizing the Banach's fixed point theorem. Sessa [20] generalized the notion of commutativity and defined weak commutativity. Further, Jungck [6] introduced more generalized commutativity, so called compatibility and generalized some results of Singh and Singh [23] and Fisher [3]. Kaneco [9] extended the concept of weakly commuting mappings for multivalued set up and extended result of Jungck [5]. Kaneko and Sessa [10] extended the concept of compatible mappings for multivalued mappings and generalized the result of Kubiak [12]. In 1998, Jungck and Rhoades [7] extended weak compatibility in the settings of single-valued and multivalued mappings. Pant [16, 17, 18, 19] initiated the study of non compatible mappings and introduced R-weak commutativity of mappings. He also showed that for single-valued mappings pointwise R-weak commutativity is equivalent to weak compatibility at the coincidence points. Shahzad and Kamran [21] and Singh and Mishra [22] have independently extended the idea of R-weak commutativity to the settings of single and multivalued mappings.

In [22], Singh and Mishra introduced the notion of (IT)-commutativity for a hybrid pair of single-valued and multivalued mappings and showed that a pointwise R-weakly commuting hybrid pair need not be weakly compatible. However at the coincidence points pointwise R-weak commutativity for hybrid pairs is equivalent to (IT)-commutativity.

Recently, Kamaran [8] introduced the notion of T-weak commutativity for a single-valued and a multivalued mapping and showed that it is weaker condition than (IT)-commutativity and weak compatibility of hybrid pair.

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In their paper, Kaneko [9] and Kaneko and Sessa [10] have assumed a pair of single-valued and multivalued mapping which are continuous at X and could prove the existence of a coincidence point. For the existence of a common fixed point an additional hypothesis is needed. They have also remarked whether or not the continuity of two mappings is really needed in the proof.

Kubiaczyk and Mustafa Ali [11], Krzyska and Kubiaczyk [14] and many others have proved common fixed point theorems for multivalued mappings.

Asad and Ahmad [1] extended the results of Fisher [2] for multivalued mappings using condition of weak commutativity or compatibility and proved that existence of common fixed point can be achieved by the continuity of the single-valued mapping only, the continuity of the multivalued mappings are not needed.

In this paper, we improve results of Asad and Ahmad [1] by taking T-weak commutativity in place of weak commutativity or in place of compatibility, without assuming continuity of any mapping.

2. Preliminaries

Let (X, d) be a metric space and suppose that $CB(X)$ denotes the set of non-empty closed and bounded subsets of X.

For A, B in $CB(X)$ we denote

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$
$$D(x,A) = \inf\{d(x,a) : a \in A\}$$
$$H(A,B) = \max\{\sup\{D(a,B) : a \in A\}, \sup\{D(A,b) : b \in B\}\}.$$

Kuratowski [13] showed that $(CB(X), H)$ is a metric space with the distance function H, moreover $(CB(X), H)$ is complete in the event that (X, d) is complete.

Lemma 1 ([15]). *Let $A, B \in CB(X)$, then for $\varepsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a,b) \le H(A,B) + \varepsilon$. If A and B are compact then one can find $b \in B$ such that $d(a,b) \le H(A,B)$.*

Definition 1 ([9]). Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair $\{F, T\}$ is said to be weakly commuting if for each $x \in X$, $TF(x) \in CB(X)$ and $H(FTx, TFx) \le D(Tx, Fx)$.

Definition 2 ([10]). Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair $\{F, T\}$ is said to be compatible if and only if $TFx \in CB(X)$ for each $x \in X$ and $H(FTx_n, TFx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to M \in CB(X)$ and $Tx_n \to t \in M$.

Definition 3 ([7]). The mapping $T : X \to X$ and $F : X \to CB(X)$ are weakly compatible if they commute at their coincidence points that is if $FTu = TFu$ whenever $Tu \in Fu$.

Definition 4 ([4, 22]). The mapping $T : X \to X$ and $F : X \to CB(X)$ are said to be (IT)-commuting at $x \in X$ if $TFx \subseteq FTx$.

Definition 5 ([21])**.** The mappings $T : X \to X$ and $F : X \to CB(X)$ are said to be R-weakly commuting if, for given $x \in X$, $TFx \in CB(X)$ and there exists some positive real number R such that $H(TFx, FTx) \le Rd(Tx, Fx)$.

Definition 6 ([8])**.** Let $T : X \to X$ and $F : X \to CB(X)$ the hybrid pair $\{T, F\}$ is said to be T-weakly commuting at $x \in X$ if $TTx \in FTx$.

Example 1. Let $X = [0, \infty)$ with the usual metric. Define $T : X \to X$ and $F : X \to CB(X)$ by $Tx = 3x$ and $Fx = [0, 3 + 3x]$ for all $x \in X$. Then for all $x \in X$, $Tx \in Fx$, $TTx = 9x \in [0, 3 + 9x] = FTx$. Therefore the pair $\{T, F\}$ is T-weakly commuting but not (IT)-commuting because $TFx = [0, 9 + 9x] \nsubseteq FTx = [0, 3+9x]$. Also note that T and F are not weakly compatible. Moreover, if $\{x_n\}$ is a sequence in X such that $x_n \to 1$. Then $\lim_{n\to\infty} Tx_n = 3 \in [0, 6] = \lim_{n\to\infty} Fx_n$ and $\lim_{n\to\infty} H(TFx_n,\ FTx_n) = 6$. Therefore the mappings F and T are not compatible.

Remark 1 ([8])**.**

(i) Let $T : X \to X$ and $F : X \to CB(X)$. The hybrid pair $\{T, F\}$ is (IT)-commuting at the coincidence points implies that it is T-weakly commuting but T-weakly commuting hybrid pair is neither (IT)-commuting nor weakly compatible in general.

(ii) If F is single-valued mapping then T-weak commutativity at the coincidence points is equivalent to the weak compatibility.

(iii) It is known [17] that pointwise R-weak commutativity is minimal condition for the existence of fixed point.

3. Main Result

Theorem 1. *Let (X, d) be a complete metric space, $T : X \to X$ and F, $G : X \to CB(X)$, satisfying*

(3.1) $F(X) \cup G(X) \subseteq T(X)$;

(3.2) *the pairs $\{F, T\}$ and $\{G, T\}$ are T-weakly commuting at their coincidence points;*

(3.3) $H(Fx, Gy) \le \alpha \dfrac{[D(Fx, Ty)]^2 + [D(Gy, Tx)]^2}{D(Fx, Ty) + D(Gy, Tx)} + \beta d(Tx, Ty)$,

$x \ne y$, $Fx \ne Fy$, $Gx \ne Gy$ for all $x, y \in X$, $\alpha, \beta \ge 0$, $2\alpha + \beta < 1$, whenever $D(Fx, Ty) + D(Gy, Tx) \ne 0$ and $H(Fx, Gy) = 0$, whenever $D(Fx, Ty) + D(Gy, Tx) = 0$.

Then there exists a point z in X such that $z = Tz \in Fz \cap Gz$.

Proof. Assume $\theta = \frac{\alpha+\beta}{1-\alpha}$. Let $x_0 \in X$ and y_1 be an arbitrary point in Fx_0. Choose $x_1 \in X$ such that $y_1 = Tx_1$. This is possible as $F(X) \subseteq T(X)$. By Lemma 1, we can find $y_2 \in Gx_1$ such that

$$d(y_1, y_2) \le H(Fx_0, Gx_1) + \frac{1-\alpha}{1+\alpha}\theta.$$

Choose $x_2 \in X$ such that $y_2 = Tx_2$. This is also possible as $G(X) \subseteq T(X)$. Also we can find $y_3 \in Fx_2$ such that

$$d(y_2, y_3) \leq H(Fx_2, Gx_1) + \frac{1-\alpha}{1+\alpha}\theta^2.$$

Inductively, having selected $y_{2n} = Tx_{2n} \in Gx_{2n-1}$, choose $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$ such that

$$d(y_{2n+1}, y_{2n}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n}.$$

Then having selected y_{2n+1}, choose $y_{2n+2} = Tx_{2n+2} \in Gx_{2n+1}$ such that

$$d(y_{2n+1}, y_{2n+2}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n+1}.$$

Since conditions (3.1) and (3.3) are similar to that of Asad and Ahmad [1]. So as proved in [1], we can prove that $\{y_n\} = \{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that $y_n \to z$ as $n \to \infty$. Since $F(X) \subseteq T(X)$, there exists a point $p \in X$ such that $Tp = z$. By (3.3), we have

$$D(Tp, Fp) \leq D(Tp, Gx_{2n-1}) + H(Gx_{2n-1}, Fp),$$

$$D(Tp, Fp) \leq D(Tp, Gx_{2n-1}) + \\ + \alpha \frac{[D(FpTx_{2n-1})]^2 + [D(Gx_{2n-1}, Tp)]^2}{D(Fp, Tx_{2n-1}) + D(Gx_{2n-1}, Tp)} + \beta d(Tp, Tx_{2n-1}),$$

$$D(Tp, Fp) \leq D(Tp, Gx_{2n-1}) + \\ + \alpha[D(Fp, Tx_{2n-1}) + D(Gx_{2n-1}, Tp)] + \beta d(Tp, Tx_{2n-1}).$$

On letting $n \to \infty$, we get

$$D(z, Fp) \leq \alpha \cdot D(Fp, z),$$

a contradiction. Therefore $z \in Fp$ that is $z = Tp \in Fp$. So $p \in X$ is a coincidence point of F and T.

Similarly since $G(X) \subseteq T(X)$, there exists a point $q \in X$ such that $Tq = z$. By (3.3), we have

$$D(Tq, Gq) \leq D(Tq, Fx_{2n}) + H(Fx_{2n}, Gq),$$

$$D(Tq, Gq) \leq D(Tq, Fx_{2n}) + \alpha \frac{[D(Fx_{2n}, Tq)]^2 + [D(Gq, Tx_{2n})]^2}{D(Fx_{2n}, Tq) + D(Gq, Tx_{2n})} + \beta d(Tx_{2n}, Tq),$$

$$D(Tq, Gq) \leq D(Tq, Fx_{2n}) + \alpha[D(Fx_{2n}, Tq) + D(Gq, Tx_{2n})] + \beta d(Tx_{2n}, Tq).$$

On letting $n \to \infty$, we get

$$D(z, Gq) \leq \alpha \cdot D(z, Gq),$$

a contradiction. Therefore $z \in Gq$ that is $Tq \in Gq$. So $q \in X$ is a coincidence point of G and T.

Since the hybrid pair $\{T, F\}$ is T-weakly commuting at coincidence point $p \in X$. Thus $TTp \in FTp$ that is $Tz \in Fz$. Similarly T-weakly commutativity of $\{T, G\}$ at coincidence point $q \in X$ gives $Tz \in Gz$. By (3.3), we have

$$d(Tx_{2n}, Tz) \leq H(Gx_{2n-1}, Fz),$$

$$d(Tx_{2n}, Tz) \leq \alpha \frac{[D(FzTx_{2n-1})]^2 + [D(Gx_{2n-1}, Tz)]^2}{D(Fz, Tx_{2n-1}) + D(Gx_{2n-1}, Tz)} + \beta d(Tz, Tx_{2n-1}),$$

$$d(Tx_{2n}, Tz) \leq \alpha[D(Fz, Tx_{2n-1}) + D(Gx_{2n-1}, Tz)] + \beta d(Tz, Tx_{2n-1}).$$

On letting $n \to \infty$, we get

$$d(z, Tz) \leq (2\alpha + \beta) d(z, Tz),$$

a contradiction giving there by $z = Tz$. Thus we have shown that $z = Tz \in Fz \cap Gz$. This completes the proof. □

Remark 2. (i) For $\alpha = 0$, we get an extension of the well known Banach fixed point theorem.
(ii) For $\beta = 0$, we get a new result.

Remark 3. Kaneko [9] and Kaneko and Sessa [10] assumed the continuity of both single valued and multivalued mappings. They have questioned however whether the continuity of both the mappings is really needed in the proof. In our Theorem 1, we have shown that existence of a common fixed point can be achieved without assuming continuity of any mapping.

Remark 4. We improve Theorem 1 and Theorem 3.2 of Asad and Ahmad [1] by relaxing weak commutativity to T-weak commutativity and compatibility to T-weak commutativity respectively. We also remove condition of continuity of any mapping.

Remark 5 ([1])**.** The condition in the hypothesis "$x \neq y$, $Fx \neq Fy$, $Gx \neq Gy$" is necessary since the Theorem 1 fails for F and G taken as constant mappings.

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