Mathematica Moravica
Vol. **1** (1997), 109–126

Extensions of Hardy-Littlewood-Pólya and Karamata Majorization Principles

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Abstract. The following main result is proved: Let $J \subset \mathbb{R}$ be an open interval and let $x_i, y_i \in J$ $(i = 1, \dots, n)$ be real numbers such that fulfilling

$$x_1 \geq \cdots \geq x_n, \qquad y_1 \geq \cdots \geq y_n. \tag{11}$$

Then, a necessary and sufficient condition in order that

$$\sum_{i=1}^{n} f(x_i) \geq 2 \sum_{i=1}^{n} f(y_i) - n \max\{f(a), f(b), g(f(a), f(b))\} \tag{A}$$

holds for every general convex function $f : J \to \mathbb{R}$ which is in contact with function $g : f(J)^2 \to \mathbb{R}$ and for arbitrary $a, b \in J$ $(a \geq x_i \geq b$ for $i = 1, \dots, n)$, is that

$$\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \quad (k = 1, \dots, n-1), \qquad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i. \tag{12}$$

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1991 *Mathematics Subject Classification.* Primary: 49B35, 90C30; Secondary: 90C48.
Key words and phrases. General convex functions, Convex functions, Inequalities, Majorizatioin principle, Hardy-Littlewood-Pólya majorization principle, Karamata theorem, Characterization of majorization.

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