YU ISSN 1450-5932

MATHEMATICA MORAVICA

Special Vol.(2003)
MONOGRAPH

n-GROUPS IN THE LIGHT OF THE NEUTRAL OPERATIONS

By

Janez Ušan

ELECTRONIC VERSION
2006

UNIVERSITY OF KRAGUJEVAC
Technical Faculty of Čačak

Janez Ušan
Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad, Serbia and Montenegro

Balzakova 25
21000 Novi Sad, Serbia and Montenegro

Referees: Prof. Dr Mališa Žižović
Prof. Dr Ljubiša Kočinac

AMS Mathematics Subject Classification 2000. Primary: 20N15
Keywords: n-semigroup, n-quasigroup, n-group, {i,j}-neutral operation on n-groupoid.

PREFACE

As a generalization of the notion of a group, the notion of an n-group was introduced by W. Dornte in 1928 (Chapter I-1). In 1940 E. L. Post published an extensive study of n-groups in which the well-known Post's Coset Theorem appeared. M. Hosszú (1963) and L. M. Gluskin (1965) described n-groups for $n \geq 3$ using one group, one automorphism of this group and a constant. A description of n-groups ($n \geq 3$) as algebras of the type $\langle n, 1\rangle$ with laws was obtained by B. Gleichgewicht and K. Glazek in 1967 and several such descriptions can be found in the papers [Celakoski 1977], [Dudek, Glazek, Gleichgewicht 1977] and [Dudek 1995]. A significant contribution to the theory of polyadic structures gave the group of algebraists from Skopje led by Ǵ. Čupona. Categories of n-groups were considered by J. Michalski (for example in [Michalski 1979]).Books devoted to n-groups were written by W. A. Dudek and S. A. Rusakov ([Dudek 1990], [Rusakov 1992]).

This text is as an attempt to systematize the results concerning two concepts related to the theory of n-ary structures. The first one is the concept of $(n-2)$-ary $\{i, j\}$-neutral operation in n-groupoids (Chapter II-2), while the second is the concept of $(n-1)$-ary inverse operation in n-groups, which, in fact, can be obtained using the neutral operation and a certain superposition of the basic n-group operation (Chapter III-1).

While a groupoid contains at most one neutral element, for $n > 2$ there are both n-groups without neutral elements and n-groups in which all the elements are neutral. In contrast to this, as for $n = 2$, an n-groupoid can have at most one $(n-2)$-ary $\{i, j\}$-neutral operation (Chapter II-2). Also, reduced to the binary case this concept gives the standard neutral element of a groupoid.

For each $n \geq 2$, every n-group has both an $(n-2)$-ary $\{1, n\}$-neutral operation and $(n-1)$-ary inverse operation, which in the binary case becomes the usual inverse (Chapter III-1).

Establishing of two considered concepts provide several applications. So, the $\{1, n\}$-neutral operation of an n-group has an important role in describing of all nHG-algebras associated to the given n-group (Chapter IV). Similarly, using these two operations a description of super-associative algebras with n-quasigroup operations is presented in Chapter XI. In Chapter III n-groups are characterized as algebras $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$ with the mutually independent laws.

The results systematized in this text have been published in [Ušan 1988-2002], [Ušan, Žižović 1997-2002], [Ušan, Galić 2000, 2001] and [Žižović 1998].

The author is grateful to Branimir Šešelja, Dragan Acketa, Milan Grulović, Miloš Kurilić and Ljubiša Kočinac for great help in translation of this text in English.

Novi Sad, May 2003 Janez Ušan

A remark for the electronic version - 2005

In the electronic version 2005 there is a result and some new references. Further on, in some parts of the text minor corrections are made.

Novi Sad, November 2005 Janez Ušan

P.S. In Ch. XVI of the hard version $(n, m)-$groups and $NP-$polyagroups are described shortly (in only 6 pages, without proofs). In the survey article [Ušan 2005/1] $(n, m)-$groups are described in the light of the neutral operations (in 50 pages) and in [Ušan 2005/2] the results concerning $NP-$polyagroups are systematized (in 30 pages).

Reference [Gal'mak 2003] is a new book of $n-$groups.

A remark for the electronic version - 2006

In the electronic version 2006 there is *Appendix*3 and some new references. Further on, in some parts of the text minor corrections are made.

Novi Sad, September 2006 Janez Ušan

Chapter I

IN FEW WORDS ON $n-$GROUPS

1 Notion and examples

1.1. Definitions: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then: 1) we say that (Q, A) is an $n-$***semigroup** *iff for every $i, j \in \{1, \dots, n\}$, $i < j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})^1$$

*(:$< i, j >$-associative law); 2) we say that (Q, A) is an $n-$***quasigroup** *iff for every $i \in \{1, \dots, n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the following equality holds $A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$; and 3) we say that (Q, A) is a Dörnte $n-$***group** *(briefly: $n-$group) iff (Q, A) is an $n-$semigroup and an $n-$quasigroup as well.*

1.2. Remark: *A notion of an $n-$group was introduced by W. Dörnte (inspired by E. Noether) in [Dörnte 1928] as a generalization of the notion of a group.*

1.3. Example: *Let $(Q, \cdot)$ be a group. Let also*

$$A(x_1^n) \overset{def}{=} x_1 \cdot \ldots \cdot x_n$$

for each $x_1^n \in Q$. Then (Q, A) is an $n-$group.

Sketch of the prof.

a) $$\begin{aligned} A(A(x_1^n), x_{n+1}^{2n-1}) &= (x_1 \cdot \ldots \cdot x_n) \cdot x_{n+1} \cdot \ldots \cdot x_{2n-1} \\ &= x_1 \cdot \ldots \cdot x_{i-1} \cdot (x_i \cdot \ldots \cdot x_{i+n-1}) \cdot x_{i+n} \cdot \ldots \cdot x_{2n-1} \\ &= A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}). \end{aligned}$$

[1]About the expression a_p^q see Appendix I.

b) $A(a_1^{i-1}, x, a_i^{n-1}) = a_n \Longleftrightarrow (a_1 \cdot \ldots \cdot a_{i-1}) \cdot x \cdot (a_i \cdot \ldots \cdot a_{n-1}) = a_n$. □

1.4. Example: *Let* $(\{1,2,3,4\}, \cdot)$ *be the Klein group: Tab. 1. Further on, let* φ *be the permutation of the set* $\{1,2,4,3\}$ *defined in the following way*

$$\varphi \stackrel{def}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Tab. 1

Let also

$$A(x,y,z) \stackrel{def}{=} x \cdot \varphi(y) \cdot z \cdot 2$$

for all $x, y, z \in \{1,2,3,4\}$. *Then* $(\{1,2,3,4\}, A)$ *is a 3–group.*

Sketch of the prof.

a) $\varphi^2 (= \varphi \circ \varphi) = I,\ \varphi(2) = 2$ and $\varphi \in Aut(\{1,2,3,4\}, \cdot)$.

b)
$$\begin{aligned} A(A(x,y,z),u,v) &= (x \cdot \varphi(y) \cdot z \cdot 2) \cdot \varphi(u) \cdot v \cdot 2 \\ &= x \cdot (\varphi(y) \cdot z \cdot 2 \cdot \varphi(u)) \cdot v \cdot 2 \\ &\stackrel{a)}{=} x \cdot \varphi(y \cdot \varphi(z) \cdot u \cdot 2) \cdot v \cdot 2 \\ & A(x, A(y,z,u), v); \\ A(A(x,y,z),u,v) &= (x \cdot \varphi(y) \cdot z \cdot 2) \cdot \varphi(u) \cdot v \cdot 2 \\ &= x \cdot \varphi(y) \cdot (z \cdot 2 \cdot \varphi(u) \cdot v) \cdot 2 \\ & A(x, y, A(z,u,v)). \end{aligned}$$

c) $A(a,b,c) = d \Longleftrightarrow a \cdot \varphi(b) \cdot c \cdot 2 = d$. □

1.5. Example: *Let* $p_1 \stackrel{def}{=} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $p_2 \stackrel{def}{=} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ *and* $p_3 \stackrel{def}{=} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$; *Figure 1. Let also*

$$A(x_1^3) \stackrel{def}{=} x_1 \circ x_2 \circ x_3$$

for all $x_1^3 \in \{p_1, p_2, p_3\}$, *where* $f \circ g$ *is the composition of permutations. Then* $(\{p_1,p_2,p_3\}, A)$ *is a* 3−*group. [*p_1, p_2 *and* p_3 *are odd permutations!]*

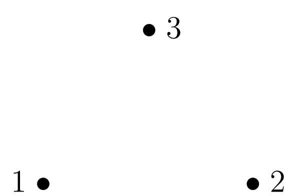


Figure 1

Remark: *This example is also in Chemistry–about ammonia* (NH_3)*. See, for example, [Dudek 1990].*

1.6.Remark: *More examples see [Gal'mak, Vorob'ev 1998] and [Gal'mak 2003].*

2 Hosszú–Gluskin Theorem

2.1. Theorem *[Hosszú 1963, Gluskin 1965]: For every* $n-$*group* (Q, A), $n \geq 3$, *there is an algebra* $(Q, \{\cdot, \varphi, b\})$ *[of the type* $< 2, 1, 0 >$*] such that the following statements hold: (1)* $(Q, \cdot)$ *is a group; (2)* $\varphi \in Aut(Q, \cdot)$; *(3)* $\varphi(b) = b$; *(4) for every* $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$; *and (5) for every* $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$.

See, also Example 1.4. In detail in Chapter IV.

3 Post's Coset Theorem

3.1. Theorem *[Post 1940]: For every* $n-$*group* (Q, A), $n \geq 3$, *there is a group* $(\overline{Q}, \cdot)$ *and its normal subgroup* $(H, \cdot)$ *such that: 1)* $Q \in \overline{Q}/H$; *2) the factor–group* $(\overline{Q}/H, \Box)$ *[of the group* $(\overline{Q}, \cdot)$ *over* H*] is a finite cyclic group, and* $|\overline{Q}/H| \,|(n-1)$; *and 3) for every* $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \ldots \cdot x_n$.

See, also Example 1.5. The proof see in Appendix 3.

4 $n-$groups as algebras of the type $< n, n-1, n-2 >$ with laws

4.1. Theorem *[Ušan 1997/2]: Let* $n \geq 2$ *and let* (Q, A) *be an* $n-$*groupoid. Then the following statements are equivalent:* (*i*) (Q, A) *is an* $n-$*group;* (*ii*) *there are mappings* $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q, \{A,^{-1}, \mathbf{e}\})$ *[of the type* $< n, n-1, n-2 >$*]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) *there are mappings* $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q, \{A,^{-1}, \mathbf{e}\})$ *[of the type* $< n, n-1, n-2 >$*]*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

The case $n = 2$ is described in [Dickson 1905] (cf. [Clifford, Preston 1964]). In detail Chapter III.

4.2. Remark: $\mathbf{e}$ *is an* $\{1, n\}-$*neutral operation of* $n-$*groupoid* (Q, A) *iff algebra* $(Q, \{A, \mathbf{e}\})$ *[of the type* $< n, n-2 >$*] satisfies the laws* (b) *and* $(\overline{b})$ *from 4.1 [Ušan 1988]. Operation* $^{-1}$ *from 4.1 [*$(c), (\overline{c})$*] is a generalization of the inverse operation in a group [Ušan 1994]. In details in Chapter II and in Chapter III.*

Chapter II

TWO GENERALIZATIONS OF A NEUTRAL ELEMENT OF A GROUPOID

1 Neutral element of $n-$groupoid

1.1. Definition: *Let (Q, A) be an $n-$groupoid and let $n \geq 2$. Then, $e \in Q$ is a* **neutral element** *of the $n-$groupoid (Q, A) iff for all $i \in \{1, \ldots, n\}$ and for all $x \in Q$ the following equality holds*

(1) $$A(\overset{i-1}{e}, x, \overset{n-i}{e}) = x.$$

1.2. Theorem *[Čupona, Trpenovski 1961]: Let (Q, A) be an $n-$semigroup and $n \geq 3$. Then, (Q, A) has a neutral element iff there is a semigroup $(Q, \cdot)$ with a neutral element such that, for every $x_1^n \in Q$, the following equality holds*

$$A(x_1^n) = x_1 \cdot \ldots \cdot x_n.$$

Sketch of a part of the proof.

Let

(2) $$x \cdot y \overset{def}{=} A(x, y, \overset{n-2}{e}).$$

$$\begin{aligned} 1)\quad (x \cdot y) \cdot z &= A(A(x, y, \overset{n-2}{e}), z, \overset{n-2}{e}) = A(x, y, A(\overset{n-2}{e}, z, e), \overset{n-3}{e}) \\ &= A(x, y, A(z, \overset{n-2}{e}, e), \overset{n-3}{e}) = A(x, A(y, z, \overset{n-2}{e}), e, \overset{n-3}{e}) \\ &= A(x, A(y, z, \overset{n-2}{e}), \overset{n-2}{e}) = x \cdot (y \cdot z). \end{aligned}$$

$$\begin{aligned} 2)\quad x \cdot e &\overset{(2)}{=} A(x, e, \overset{n-2}{e}) = A(x, \overset{n-1}{e}) \overset{(1)}{=} x, \\ e \cdot x &\overset{(2)}{=} A(e, x, \overset{n-2}{e}) \overset{(1)}{=} x. \end{aligned}$$

$$
\begin{aligned}
3)\quad A(x_1^n) &= A(A(x_1^n), \overset{n-1}{e}) = A(x_1, A(x_2^n, e), \overset{n-2}{e}) \\
&\overset{(2)}{=} x_1 \cdot A(x_2^n, e) = x_1 \cdot A(A(x_2^n, e), \overset{n-1}{e}) \\
&= x_1 \cdot A(x_2, A(x_3^n, \overset{2}{e}), \overset{n-2}{e}) = x_1 \cdot x_2 \cdot A(x_3^n, \overset{2}{e}) \\
&\quad - - - - - - - - - - - - - - - - - - \\
&\quad - - - - - - - - - - - - - - - - - - \\
&= x_1 \cdot \ldots \cdot x_{n-2} \cdot A(x_{n-1}^n, \overset{n-2}{e}) \\
&= x_1 \cdot \ldots \cdot x_{n-2} \cdot A(A(x_{n-1}^n, \overset{n-2}{e}), \overset{n-1}{e}) \\
&= x_1 \cdot \ldots \cdot x_{n-2} \cdot A(x_{n-1}, A(x_n, \overset{n-1}{e}), \overset{n-2}{e}) \\
&= x_1 \cdot \ldots \cdot x_{n-2} \cdot x_{n-1}, \cdot A(x_n, \overset{n-1}{e}) \\
&= x_1 \cdot \ldots \cdot x_{n-2} \cdot x_{n-1}, \cdot x_n.
\end{aligned}
$$

See, also [Čupona 1969]. □

By Def. 1.1. from Chapter I and Th. 1.2. from Chapter II, we conclude that the following proposition holds:

1.3. Corollary: *Let (Q, A) be an $n-$group and $n \geq 3$. Then, (Q, A) has a neutral element iff there is a group $(Q, \cdot)$ such that, for every $x_1^n \in Q$, the following equality holds*

$A(x_1^n) = x_1 \cdot \ldots \cdot x_n$.

Cf. [Belousov 1972] and [Kurosh 1974].

Furthermore, the following two propositions hold.

1.4. Proposition: *If $n \geq 3$, then there exists an $n-$group (Q, A) such that every $a \in Q$ is its neutral element.* [1]

Proof. By Example 1.3 from Chapter I if $(Q, \cdot)$ is a Klein's group and $n = 3$. □

1.5. Proposition: *If $n \geq 3$, then there exists an $n-$group (Q, A) such that no $a \in Q$ is its neutral element.*

[1] See, also [Žižović, Kočinac 1988].

Proof. By Example 1.4 from Chapter I. □

2 $\{i,j\}-$neutral operations of $n-$groupoids

2.1. Definitions *[Ušan 1988]: Let $n \geq 2$ and let (Q,A) be an $n-$groupoid. Further on, let $\mathbf{e}_L, \mathbf{e}_R$ and $\mathbf{e}$ be mappings of the set Q^{n-2} into the set Q. Let also $\{i,j\} \subseteq \{1,\ldots,n\}$ and $i<j$. Then:*

1) $\mathbf{e}_L$ is a **left** $\{i,j\}-$**neutral operation** *of the $n-$groupoid (Q,A) iff the followingg formula is satisfied*

(l) $(\forall a_t \in Q)_1^{n-2}(\forall x \in Q)A(a_1^{i-1}, \mathbf{e}_L(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x;$

2) $\mathbf{e}_R$ is a **right** $\{i,j\}-$**neutral operation** *of the $n-$groupoid (Q,A) iff the followingg formula holds*

(r) $(\forall a_t \in Q)_1^{n-2}(\forall x \in Q)A(a_1^{i-1}, x, a_i^{j-2}, \mathbf{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = x;$ *and*

3) $\mathbf{e}$ is an $\{i,j\}-$**neutral operation of the** $n-$**groupoid** (Q,A) *iff the followingg formula holds*

(n) $(\forall a_t \in Q)_1^{n-2}(\forall x \in Q)(A(a_1^{i-1}, \mathbf{e}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x \wedge$

$$A(a_1^{i-1}, x, a_i^{j-2}, \mathbf{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x).$$

2.2. Remark: *An $\{i,j\}-$neutral (left, right) operation of an $n-$groupoid is a generalization of the notion of a neutral (left, right) element of a groupoid. Namely, for $n=2$, $\mathbf{e}(a_1^{n-2})$ [$= \mathbf{e}(\emptyset)$] is a neutral (left, right) elememt of the groupoid* (Q,A) $(: n-2=0,\ i=1,\ j=2)$. *Besides, $\mathbf{e}$ is a nulary operation in the set Q.*

2.3. Proposition *[Ušan 1988]: Let (Q, A) be an n-groupoid and $n \geq 2$. Also let $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i < j$. Then there is* **at most one** *$\{i, j\}-$neutral operation of (Q, A).*

Proof. Suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are $\{i, j\}-$neutral operations of an n-groupoid (Q, A). Then, by Def. 2.1, for every sequence a_1^{n-2} over Q the following equalities hold

$$A(a_1^{i-1}, \mathbf{e}_1(a_1^{n-2}), a_i^{j-2}, \mathbf{e}_2(a_1^{n-2}), a_{j-1}^{n-2}) = \mathbf{e}_2(a_1^{n-2}) \text{ and}$$

$$A(a_1^{i-1}, \mathbf{e}_1(a_1^{n-2}), a_i^{j-2}, \mathbf{e}_2(a_1^{n-2}), a_{j-1}^{n-2}) = \mathbf{e}_1(a_1^{n-2}),$$

whence we conclude that $\mathbf{e}_1 = \mathbf{e}_2$. □

2.4. Proposition *[Ušan 1988]: Let (Q, A) be an $n-$groupoid, $n \geq 2$, $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i < j$. Then: if $\mathbf{e}_L$ is a* **left** *$\{i, j\}-$neutral operation of (Q, A) and $\mathbf{e}_R$ is a* **right** *$\{i, j\}-$neutral operation of (Q, A), then $\mathbf{e}_L = \mathbf{e}_R$ and $\mathbf{e} = \mathbf{e}_L = \mathbf{e}_R$ is an $\{i, j\}-$neutral operation of (Q, A).*

Proof. By Def. 2.1, we conclude that for every sequence a_1^{n-2} over Q the following equalities hold

$$A(a_1^{i-1}, \mathbf{e}_L(a_1^{n-2}), a_i^{j-2}, \mathbf{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = \mathbf{e}_R(a_1^{n-2}) \text{ and}$$

$$A(a_1^{i-1}, \mathbf{e}_L(a_1^{n-2}), a_i^{j-2}, \mathbf{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = \mathbf{e}_L(a_1^{n-2}),$$

whence we conclude that $\mathbf{e}_L = \mathbf{e}_R$. □

2.5. Proposition *[Ušan 1997/2]: Let (Q, A) be an $n-$groupoid and let $n \geq 2$. Further on, let the following statements hold:*
(i) The $< 1, n >$-associative law holds in (Q, A);
(ii) For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, there is **at least one** *$x \in Q$ such that the equality $A(a, a_1^{n-2}, x) = b$ holds; and*
(iii) For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, there is **at least one** *$y \in Q$ such that the equality $A(y, a_1^{n-2}, a) = b$ holds.*[2]

[2]For $n = 2$ it is a group. For $n \geq 3$ there exist (Q, A) satisfying conditions (i) – (iii) which are not $n-$groups.

Then (Q,A) *has a* $\{1,n\}-$*neutral operation.*

Proof. Firstly we prove the following statements:
1° (Q,A) has a left $\{1,n\}-$neutral operation; and
2° (Q,A) has a right $\{1,n\}-$neutral operation.

The proof of 1° :
By (iii), for every sequence a_1^{n-2} over Q and for every $a \in Q$ there is at least one $\mathbf{e}_L^{(a)}(a_1^{n-2}) \in Q$ such that the following equality holds

(1) $$A(\mathbf{e}_L^{(a)}(a_1^{n-2}), a_1^{n-2}, a) = a$$

On the other hand, by (ii), for every $b \in Q$ and for every sequence k_1^{n-2} over Q there is at least one $k \in Q$ such that the following equality holds

(2) $$b = A(a, k_1^{n-2}, k).$$

By (1), (2) and (i), we conclude that the following series of equalities holds

$$\begin{aligned} A(\mathbf{e}_L^{(a)}(a_1^{n-2}), a_1^{n-2}, b) &\overset{(2)}{=} A(\mathbf{e}_L^{(a)}(a_1^{n-2}), a_1^{n-2}, A(a, k_1^{n-2}, k)) \\ &\overset{(i)}{=} A(A(\mathbf{e}_L^{(a)}(a_1^{n-2}), a_1^{n-2}, a), k_1^{n-2}, k) \\ &\overset{(1)}{=} A(a, k_1^{n-2}, k) \\ &\overset{(2)}{=} b, \end{aligned}$$

whence we conclude that for every $b \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$A(\mathbf{e}_L^{(a)}(a_1^{n-2}), a_1^{n-2}, b) = b,$$

i.e., that (Q,A) has the left $\{1,n\}-$neutral operation $\mathbf{e}_L$.

Similarly, it is possible to prove that there is a right $\{1,n\}-$neutral operation $\mathbf{e}_R$ in (Q,A) [:2°].

Finally, by Proposition 2.4, we conclude that there is an $\{1,n\}-$neutral operation $\mathbf{e}$ [$= \mathbf{e}_L = \mathbf{e}_R$]. □

By Proposition 2.5 and Def. 1.1. from Chaper I, we conclude that the following proposition holds:

2.6. Theorem *[Ušan 1988]: Every $n-$group $(n \geq 2)$ has an $\{1, n\}-$neutral operation.*

In Chapter V, $n-$groups with $\{i, j\}-$neutral operations for $\{i, j\} \neq \{1, n\}$ are described.

Chapter III

n-GROUPS AS ALGEBRAS OF THE TYPE $< n, n-1, n-2 >$ WITH LAWS

1 One generalization of an inverse operation in the group

1.1. Proposition: *Let* (Q, A) *be an* $< 1, n >$*-associative* n*-groupoid,* $\mathbf{e}$ *its* $\{1, n\}$*-neutral operation, and* $n \geq 2$*. Further on, let* E *be an* $\{1, 2n-1\}$*-neutral operation of the* $(2n-1)$*-groupoid* $(Q, \overset{2}{A})$*, where* $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$*. Also let*

(1) $$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}).[1]$$

Then, for every $a_1^{n-2}, a, x \in Q$ *the following equalities hold*

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x,$$

$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x,$$

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \text{ and}$$

$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}).[2]$$

Sketch of the proof.

1) $$\begin{aligned} A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) &= A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, x) \\ &= \overset{2}{A}((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a, a_1^{n-2}, x) \\ &= x. \end{aligned}$$

[1] [Ušan 1994].

[2] For $n = 2$, $a^{-1} = \mathsf{E}(a)$; a^{-1} is the inverse element of the element a with respect to the neutral element $\mathsf{e}(\emptyset)$ of the group (Q, A).

$$2)\quad A(A(x,a_1^{n-2},a),a_1^{n-2},(a_1^{n-2},a)^{-1}) = \overset{2}{A}(x,a_1^{n-2},a,a_1^{n-2},(a_1^{n-2},a)^{-1}) = x.$$

$$3)\quad A((a_1^{n-2},a)^{-1},a_1^{n-2},a) = A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2}))) \overset{1)}{=} \mathbf{e}(a_1^{n-2}).$$

$$4)\quad A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}) = A(A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),a_1^{n-2},(a_1^{n-2},a)^{-1}) \overset{2)}{=} \mathbf{e}(a_1^{n-2}).$$

□

1.2. Proposition *[Ušan 1997/2]: Let (Q,A) be an $n-$groupoid and let $n \geq 2$. Further on, let the following statements hold:*

(i) The $< 1, n >$-associative law holds in (Q,A);

(ii) For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, there is **at least one** *$x \in Q$ such that the equality $A(a,a_1^{n-2},x) = b$ holds; and*

(iii) For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, there is **at least one** *$y \in Q$ such that the equality $A(y,a_1^{n-2},a) = b$ holds.*

Then there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into Q such that for every $a_1^{n-2},a,x \in Q$ the following equalities hold

$$(a)\qquad A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},x)) = x,$$

$$(b)\qquad A(A(x,a_1^{n-2},a),a_1^{n-2},(a_1^{n-2},a)^{-1}) = x,$$

$$(c)\qquad A((a_1^{n-2},a)^{-1},a_1^{n-2},a) = \mathbf{e}(a_1^{n-2}) \text{ and}$$

$$(d)\qquad A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}) = \mathbf{e}(a_1^{n-2}).$$

Proof. Firstly we prove the following statements:

1° (Q,A) has an $\{1,n\}$-neutral operation $\mathbf{e}$;

2° $(Q,\overset{2}{A})$, where $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n)x_{n+1}^{2n-1})$, is an $< 1, 2n-1 >$-associative $(2n-1)$-groupoid;

3° For every $a_1^{2n-1} \in Q$, there is at least one $x \in Q$ and at least one $y \in Q$

such that the following equalities

$$\overset{2}{A}(a_1^{2n-2}, x) = a_{2n-1} \text{ and } \overset{2}{A}(y, a_1^{2n-2}) = a_{2n-1}$$

hold; and

4° $(Q, \overset{2}{A})$ has an $\{1, 2n-1\}$-neutral operation E.

The proof of 1° :
By Proposition 2.5 from Chapter II.

Sketch of the proof of 2° :

$\overset{2}{A}(\overset{2}{A}(x, a_1^{n-2}, y, b_1^{n-2}, z), c_1^{n-2}, u, d_1^{n-2}, v) =$

$A(A(A(A(x, a_1^{n-2}, y), b_1^{n-2}, z), c_1^{n-2}, u), d_1^{n-2}, v) =$

$A(A(A(x, a_1^{n-2}, y), b_1^{n-2}, z), c_1^{n-2}, A(u, d_1^{n-2}, v)) =$

$A(A(x, a_1^{n-2}, y), b_1^{n-2}, A(z, c_1^{n-2}, A(u, d_1^{n-2}, v))) =$

$A(A(x, a_1^{n-2}, y), b_1^{n-2}, A(A(z, c_1^{n-2}, u), d_1^{n-2}, v)) =$

$\overset{2}{A}(x, a_1^{n-2}, y, b_1^{n-2}, \overset{2}{A}(z, c_1^{n-2}, u, d_1^{n-2}, v)).$

Sketch of the proof of 3° :

$\overset{2}{A}(a_1^{2n-2}, x) = a_{2n-1} \Longleftrightarrow A(A(a_1^n), a_{n+1}^{2n-2}, x) = a_{2n-1}$ and

$\overset{2}{A}(y, a_1^{2n-2}) = a_{2n-1} \Longleftrightarrow A(y, a_1^{n-2}, A(a_{n-1}^{2n-2})) = a_{2n-1}.$

The proof of 4° :

By 2°, 3° and Proposition 2.5 from Chapter II.

Finally by 1°, 4°, (1) and Proposition 1.1, we conclude that the equalities $(a) - (d)$ hold. □

By Proposition 1.2, Theorem 2.6 from Chapter II, Def. 2.1 from Chapter II and Def. 1.1 from Chapter I, we conclude that the following proposition holds:

1.3. Theorem *[Ušan 1988, 1994]: Let* (Q, A) *be an* n*-group and* $n \geq 2$.

Then there are mappings $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into* Q *such that for every* $a_1^{n-2}, a, x \in Q$ *the following equalities hold*

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x,$$

$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x,$$

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}),$$

$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}),$$

$$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \text{ and}$$

$$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x.$$

2 Auxiliary propositions

2.1. Proposition *[Ušan 1997/2]: Let* (Q, A) *be an* $n-$*groupoid and let* $n \geq 3$. *Then the following statements hold:*

(I) If (a) *the* $< 1, 2 >$*-associative law holds in* (Q, A), *and* (b) *for every* $a_1^{n-1}, x, y \in Q$ *the following implication holds*

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y,$$

then (Q, A) *is an n-semigroup; and*

(II) If $(\overline{a})$ *the* $< n - 1, n >$*-associative law holds in* (Q, A), *and* $(\overline{b})$ *for every* $a_1^{n-1}, x, y \in Q$ *the following implication holds*

$$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y,$$

then (Q, A) *is an n-semigroup.*

Proof. In the proof of the proposition we use **the method** of E. I. **Sokolov** from [Sokolov 1976] */* from the Theorem 1 in [Sokolov 1976]*/*.

Sketch of the proof of the statement (I):

$$A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}) = A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}) \Rightarrow$$

$$A(b_1, A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}), b_2^{n-1}) =$$

$$A(b_1, A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}), b_2^{n-1}) \overset{(a)}{\Rightarrow}$$
$$A(A(b_1, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-2}), a_{2n-1}, b_2^{n-1}) =$$
$$A(A(b_1, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-2}), a_{2n-1}, b_2^{n-1}) \overset{(b)}{\Rightarrow}$$
$$A(b_1, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-2}) = A(b_1, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-2}).$$

Similarly, it is possible to prove that the statement (II) also holds. □

2.2. Proposition: *Let (Q, A) be an $n-$groupoid and let $n \geq 3$. Also let the following statements hold:*

(i) (Q, A) is an $n-$semigroup;

(ii) For every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds

$$A(a_1^{n-1}, x) = a_n; \text{ and}$$

(iii) For every $a_1^n \in Q$ there is exactly one $y \in Q$ such that the following equality holds

$$A(y, a_1^{n-1}) = a_n.$$

Then (Q, A) is an $n-$group.

Proof. Firstly we prove the following statements:

1° *For every $i \in \{1, \ldots, n-2\}$, for every $a, b, x, y \in Q$ and for every sequence a_1^{n-3} over Q the following implications hold*

(a) $A(a, a_1^{i-1}, x, a_i^{n-3}, b) = A(a, a_1^{i-1}, y, a_i^{n-3}, b) \Rightarrow x = y$ *and*

(b) $A(a, a_i^{n-3}, x, a_1^{i-1}, b) = A(a, a_i^{n-3}, y, a_1^{i-1}, b) \Rightarrow x = y;$

2° *For every $i \in \{1, \ldots, n-2\}$, for every $a, b, c \in Q$ and for every sequence a_1^{n-3} over Q there is at least one $x \in Q$ such that*

$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = c; \text{ and}$$

3° *(Q, A) is an $n-$quasigroup.*

Sketch of the proof of 1° :

a) $A(a, a_1^{i-1}, x, a_i^{n-3}, b) = A(a, a_1^{i-1}, y, a_i^{n-3}, b) \Rightarrow$

$A(c_{i+1}^{n-1}, A(a, a_1^{i-1}, x, a_i^{n-3}, b), c_1^i) = A(c_{i+1}^{n-1}, A(a, a_1^{i-1}, y, a_i^{n-3}, b), c_1^i) \overset{(i)}{\Rightarrow}$

$$A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, x), a_i^{n-3}, b, c_1^i) = A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, y), a_i^{n-3}, b, c_1^i) \overset{(iii)}{\Rightarrow}$$
$$A(c_{i+1}^{n-1}, a, a_1^{i-1}, x) = A(c_{i+1}^{n-1}, a, a_1^{i-1}, y) \overset{(ii)}{\Rightarrow} x = y.$$

b) $A(a, a_i^{n-3}, x, a_1^{i-1}, b) = A(a, a_i^{n-3}, y, a_1^{i-1}, b) \Rightarrow$
$$A(c_1^i, A(a, a_i^{n-3}, x, a_1^{i-1}, b), c_{i+1}^{n-1}) = A(c_1^i, A(a, a_i^{n-3}, y, a_1^{i-1}, b), c_{i+1}^{n-1}) \overset{(i)}{\Rightarrow}$$
$$A(c_1^i, a, a_i^{n-3}, A(x, a_1^{i-1}, b, c_{i+1}^{n-1})) = A(c_1^i, a, a_i^{n-3}, A(y, a_1^{i-1}, b, c_{i+1}^{n-1})) \overset{(ii)}{\Rightarrow}$$
$$A(x, a_1^{i-1}, b, c_{i+1}^{n-1}) = A(y, a_1^{i-1}, b, c_{i+1}^{n-1}) \overset{(iii)}{\Rightarrow} x = y.$$

Sketch of the proof of 2° :
$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = c \overset{1^\circ(b)}{\Longleftrightarrow}$$
$$A(c_{i+1}^{n-1}, A(a, a_1^{i-1}, x, a_i^{n-3}, b), c_1^i) = A(c_{i+1}^{n-1}, c, c_1^i) \overset{(i)}{\Longleftrightarrow}$$
$$A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, x), a_i^{n-3}, b, c_1^i) = A(c_{i+1}^{n-1}, c, c_1^i),$$
where c_1^{n-1} is an arbitrary sequence over Q. Whence, by (iii) and by (ii), we conclude that the statement 2° holds.

The proof of 3° : By $(ii), (iii), 1^\circ - (a)$ and by 2°.

Finally, by (i), 3° and by Def. 1.1 from Chapter I, we conclude that the (Q, A) is an $n-$group. □

By Proposition 2.1 and Def. 1.1 from Chapter I, we conclude that the following proposition holds:

2.3. Proposition: *Let* (Q, A) *$n-$groupoid and* $n \geq 3$. *Also let the following statements hold:*

(1) The $< 1, 2 >$*-associative [***or** $< n-1, n >$*-associative] law holds in* (Q, A); *and*

(2) (Q, A) *is an n-quasigroup.*

Then (Q, A) *is an $n-$group.*[3]

[3]See, also [Sokolov 1976] and [Dudek, Glazek, Gleichgewicht 1977].

3 Main propositions

3.1. Theorem *[Ušan 1997/2]: Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then: (Q, A) is an $n-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A,^{-1}, \mathbf{e})$ [of the type $< n, n-1, n-2 >$]*

(1) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(2) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(3) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$.

Proof. 1) $\Longrightarrow$: By Def. 1.1 from Chapter I and Theorem 1.3 from Chapter III.

2) $\Longleftarrow$: Firstly we prove the following statements:

1° For every $a_1^{n-1}, x, y \in Q$

$$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y;$$

2° (Q, A) is an $n-$semigroup;

3° $(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$;

4° $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$;

5° For every $a_1^{n-1}, x, y \in Q$

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y; \text{ and}$$

6° For every $a_1^n \in Q$ there is **exactly one** $x \in Q$ and **exactly one** $y \in Q$ such that the following equalities hold

$$A(a_1^{n-1}, x) = a_n \text{ and } A(y, a_1^{n-1}) = a_n.$$

Sketch of the proof of the statement 1° :

a) The case $n = 2$:

$$\begin{aligned} A(a, x) = A(a, y) &\Rightarrow A(a^{-1}, A(a, x)) = A(a^{-1}, A(a, y)) \\ &\overset{(1)}{\Rightarrow} A(A(a^{-1}, a), x) = A(A(a^{-1}, a), y) \\ &\overset{(3)}{\Rightarrow} A(\mathbf{e}(\emptyset), x) = A(\mathbf{e}(\emptyset), y) \overset{(2)}{\Rightarrow} x = y; \end{aligned}$$

b) The case $n > 2$:

$A(a_1^{n-2}, a, x) = A(a_1^{n-2}, a, y) \Rightarrow$
$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, \mathbf{e}(a_1^{n-2}), A(a_1^{n-2}, a, x)) =$
$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, \mathbf{e}(a_1^{n-2}), A(a_1^{n-2}, a, y)) \overset{(1)}{\Rightarrow}$
$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a), x) =$
$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a), y) \overset{(2)}{\Rightarrow}$
$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, a, x) = A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, a, y) \overset{(2)}{\Rightarrow} x = y.$

The proof of the statement 2° :
For $n \geq 3$, by 1° and Proposition 2.1. For $n = 2$:(1).

Sketch of the proof of the statement 3° :

$A(a, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = b \Rightarrow$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{2^\circ}{\Longrightarrow}$
$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(3)}{\Rightarrow}$
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(2)}{\Rightarrow}$
$\mathbf{e}(a_1^{n-2}) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(3)}{\Rightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{1^\circ}{\Longrightarrow} a = b.$

Sketch of the proof of the statement 4° :

$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = b \Rightarrow$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) =$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{2^\circ}{\Rightarrow}$
$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(3)}{\Rightarrow}$
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(2),3^\circ}{\Longrightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{1^\circ}{\Rightarrow} b = \mathbf{e}(a_1^{n-2}).$

Sketch of the proof of the statement 5° :

$A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow$
$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{2^\circ}{\Rightarrow}$
$A(x, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) = A(y, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) \overset{4^\circ}{\Rightarrow}$
$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A(y, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) \overset{3^\circ}{\Rightarrow} x = y.$

Sketch of the proof of the statement 6° :

a) $A(x, a_1^{n-2}, a) = b \overset{5^\circ}{\Leftrightarrow}$
$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{2^\circ}{\Leftrightarrow}$
$A(x, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{4^\circ}{\Leftrightarrow}$
$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{3^\circ}{\Leftrightarrow}$
$x = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1})$.

b) $A(a, a_1^{n-2}, y) = b \overset{1^\circ}{\Leftrightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, y)) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{2^\circ}{\Leftrightarrow}$
$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, y) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(3)}{\Leftrightarrow}$
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \overset{(2)}{\Leftrightarrow}$
$y = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b)$.

Finally, by 2° and 6° for $n = 2$ and $2^\circ, 6^\circ$ and Proposition 2.2 for $n \geq 3$, we conclude that (Q, A) is an $n-$group. □

3.2. Theorem *[Ušan 1997/2]: Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then there is* **at most one** *pair of mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws (1)-(3) from Theorem 3.1 hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$.*

Proof. Assume that there are mappings
$$^{-1_k} : Q^{n-1} \to Q \text{ and } \mathbf{e}_k : Q^{n-2} \to Q,\ k \in \{1, 2\},$$
such that the laws (1)-(3) from Theorem 3.1 hold in the algebras $(Q, \{A, ^{-1_1}, \mathbf{e}_1\})$ and $(Q, \{A, ^{-1_2}, \mathbf{e}_2\})$. Whence, by Th. 3.1, we conclude that the following statement holds:

1^* (Q, A) is an $n-$group.

By 1^*, by Th. 2.6 from Chapter II, and by Prop. 2.3 from Chapter II, we conclude that the following statement holds:

2^* $\mathbf{e}_1 = \mathbf{e}_2$ $(= \mathbf{e})$.

In addition, by 2^* and by (3) from Th. 3.1, we conclude that the following statement holds:

3^* for all $a \in Q$ and for every sequence a_1^{n-2} over Q, the following equalities hold

$$A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \text{ and } A((a_1^{n-2}, a)^{-1_2}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}).$$

Finally, by 1*, 3* and by Def. 1.1 from Chapter I, we conclude that for all $a \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$(a_1^{n-2}, a)^{-1_1} = (a_1^{n-2}, a)^{-1_2},$$

i.e., that $^{-1_1} = ^{-1_2}$.

3.3. Theorem *[Ušan 1997/2]: The laws (1)-(3) from Theorem 3.1 are mutually independent.*

Proof. a) The laws (1) and (2) from Th. 3.1 hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_n$, $^{-1}$ an arbitrary $(n-1)-$ary operation in Q and $\mathbf{e}(a_1^{n-2}) \overset{def}{=} c-$constant. However, the law (3) from Th. 3.1 does not hold.

b) The laws (1) and (3) from Th. 3.1 hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_1$, $\mathbf{e}(a_1^{n-2}) \overset{def}{=} c$-constant and $(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathbf{e}(a_1^{n-2})$. However, the law (2) from Th. 3.1 does not hold.

c_1) The case $n > 2$: Let $(Q, \square)$ be a group, $^{\mathbf{-1}}$ its inverse operation, and let (Q, B) be an $(n-2)-$groupoid which is not an $(n-2)-$quasigroup [for $n = 3$: $B \notin Q!$]. Then (Q, A), where

$$A(x, a_1^{n-2}, y) \overset{def}{=} x \square (B(a_1^{n-2}))^{\mathbf{-1}} \square y,$$

satisfies conditions of Proposition 2.5 from Chapter II and of Proposition 1.2 from Chapter III. Thus, there is an algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, in which the laws (2) and (3) from Th. 3.1 hold. However, the law (1) fails to hold in $(Q, \{A,^{-1}, \mathbf{e})$. Indeed, if the law (1) from Th. 3.1 holds in $(Q, \{A,^{-1}, \mathbf{e}\})$, then by the Th. 3.1 (Q, A) is an $n-$group, which contradicts the assumption that (Q, B) is not an $(n-2)-$quasigroup [for $n = 3$: $B \in Q!$].

c_2) The case $n = 2$: Let (Q, A) be a Moufang loop which is not a group, let $e = \mathbf{e}(\emptyset)$ be its neutral element and $^{-1}$ its inverse operation (cf. [Bruck

1958] or [Belousov 1967]). Then the laws (2) and (3) from Th. 3.1 hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< 2, 1, 0 >$. However, the law (1) does not hold. □

In this way, an algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$ is associated to every $n-$group (Q, A); Th. 1.3. Among laws which hold in this algebra, we point out the following ones:

(1_L) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

(1_R) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(2_L) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$,

(2_R) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$,

(3_L) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$,

(3_R) $A(a, , a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$,

(4_L) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$ and

(4_R) $A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x$.[4]

Further on, among the above laws, we choose six [for $n = 2$ four[5]] systems of three laws, as described in tables 1-6 [the laws which are represented by marked fields in Tables 1-6]. Namely, the following proposition holds:

3.4. Theorem *[Ušan 1997/2]: Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then the following statement holds:*

(a_i) (Q, A) is an $n-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$ the laws from the marked fields from the Table i, $i \in \{1, \dots, 6\}$, hold;

	L	R
1		•
2	•	
3	•	
4		

Tabl. 1

	L	R
1	•	
2		•
3		•
4		

Tabl. 2

	L	R
1	•	
2	•	
3		
4		•

Tabl. 3

---

[4] $(1) = (1_R)$, $(2) = (2_L)$ and $(3) = (3_L)$, where laws (1)-(3) are from Th. 3.1.

[5] For $n = 2$, $(1_L) = (1_R)$.

(b_i) *There is at most one pair of mappings* $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that in the algebra* $(Q, \{A, ^{-1}, \mathbf{e}\})$ *of the type* $< n, n-1, n-2 >$ *the laws table* i*,* $i \in \{1, \dots, 6\}$ *hold; and*

	L	R
1		•
2	•	
3		
4		•

Tabl. 4

	L	R
1		•
2		•
3		
4	•	

Tabl. 5

	L	R
1	•	
2		•
3		
4	•	

Tabl. 6

(c_i) *The laws from the marked fields from the Table* i*,* $i \in \{1, \dots, 6\}$*, are mutually independent.*

Proof. 1) $i = 1$: Th. 3.4 – (a_1) is a Th. 3.1, Th. 3.4 – (b_1) is a Th. 3.2, and Th. 3.4 – (c_1) is a Th. 3.3.
2) The proof of (a_i), $i \in \{2, \dots, 6\}$:
$i = 2$: Similarly to the proof of Th. 3.1; Table 1 and Table 2.
$i = 3$: Firstly we prove the following statements:

$^\circ 1$ $A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow x = y$ for all $x, y, a, a_1^{n-2} \in Q$;

$^\circ 2$ (Q, A) is an $n-$semigroup;

$^\circ 3$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$ for all $a, a_1^{n-2} \in Q$; and

$^\circ 4$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$.

The proof of $^\circ 1$: By (4_R).

The proof of $^\circ 2$: By $^\circ 1$, and by Prop. 2.1.

Sketch of the proof of $^\circ 3$:

$$\begin{aligned}\mathbf{e}(a_1^{n-2}) &\overset{(4_R)}{=} A(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1})\\ &\overset{^\circ 2}{=} A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, A(a, a_1^{n-2}(a_1^{n-2}, a)^{-1}))\\ &\overset{(2_L)}{=} A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}).\end{aligned}$$

Sketch of the proof of $^\circ 4$:

$$\begin{aligned}x &\overset{(4_R)}{=} A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1})\\ &\overset{^\circ 2}{=} A(x, a_1^{n-2}, A(a, a_1^{n-2}(a_1^{n-2}, a)^{-1}))\\ &\overset{^\circ 3}{=} A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})).\end{aligned}$$

Finally, by $^\circ 2$, $^\circ 3[= (3_R)]$, $^\circ 4[= (2_R)]$, and by Th. 3.4 – (a_2), we conclude that Th. 3.4 – (a_3) holds.

$i = 4$: For $n = 2$ the case "$i = 4$" coincides with the case "$i = 3$". Let

$n\geq 3$. Firstly we prove the following statements:

$\overline{1}$ $A(a_1^{n-2},a,x)=A(a_1^{n-2},a,y)\Rightarrow x=y$ for all $x,y,a,a_1^{n-2}\in Q$;

$\overline{2}$ (Q,A) is an $n-$semigroup;

$\overline{3}$ $A(a,a_1^{n-2},(a_1^{n-2},a)^{-1})=\mathbf{e}(a_1^{n-2})$ for all $a,a_1^{n-2}\in Q$; and

$\overline{4}$ $A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))=x$ for all $x,a_1^{n-2}\in Q$.

Sketch of the proof of $\overline{1}$:

$$\begin{aligned}
&A(a_1^{n-2},a,x)=A(a_1^{n-2},a,y)\Rightarrow\\
&A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},\mathbf{e}(a_1^{n-2}),A(a_1^{n-2},a,x))=A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},\mathbf{e}(a_1^{n-2}),A(a_1^{n-2},a,y))\Rightarrow\\
&A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),x)=A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),y)\Rightarrow\\
&A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},a,x)=A(\mathbf{e}(\overset{n-2}{a}),\overset{n-3}{a},a,y)\Rightarrow x=y.
\end{aligned}$$

The proof of $\overline{2}$: By $\overline{1}$, and by Prop. 2.1.

Sketch of the proof of $\overline{3}$:

$$\begin{aligned}
\mathbf{e}(a_1^{n-2})&\overset{(4_R)}{=}A(A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),a_1^{n-2},(a_1^{n-2},a)^{-1})\\
&\overset{\overline{2}}{=}A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}))\\
&\overset{(2_L)}{=}A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}).
\end{aligned}$$

Sketch of the proof of $\overline{4}$:

$$\begin{aligned}
x&\overset{(4_R)}{=}A(A(x,a_1^{n-2},a),a_1^{n-2},(a_1^{n-2},a)^{-1})\\
&\overset{\overline{2}}{=}A(x,a_1^{n-2},A(a,a_1^{n-2}(a_1^{n-2},a)^{-1}))\\
&\overset{\overline{3}}{=}A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})).
\end{aligned}$$

Finally, by $\overline{2}$, $\overline{3}[=(3_R)]$, $\overline{4}[=(2_R)]$, and by Th. 3.4 – (a_2), we conclude that Th. 3.4 – (a_4) holds.

$i=5$: Firstly we prove the following statements:

$\widehat{1}$ $A(a,a_1^{n-2},x)=A(a,a_1^{n-2},y)\Rightarrow x=y$ for all $x,y,a,a_1^{n-2}\in Q$;

$\widehat{2}$ (Q,A) is an $n-$semigroup;

$\widehat{3}$ $A((a_1^{n-2},a)^{-1},a_1^{n-2},a)=\mathbf{e}(a_1^{n-2})$ for all $a,a_1^{n-2}\in Q$; and

$\widehat{4}$ $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$ for all $x,a_1^{n-2}\in Q$.

The proof of $\widehat{1}$: By (4_L).

The proof of $\widehat{2}$: By $\widehat{1}$, and by Prop. 2.1.

Sketch of the proof of $\widehat{3}$:

$$\begin{aligned}\mathbf{e}(a_1^{n-2}) &\overset{(4_L)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})))\\ &\overset{\widehat{2}}{=} A(A((a_1^{n-2},a)^{-1},a_1^{n-2},a),a_1^{n-2},\mathbf{e}(a_1^{n-2}))\\ &\overset{(2_R)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},a).\end{aligned}$$

Sketch of the proof of $\widehat{4}$:

$$\begin{aligned}x &\overset{(4_L)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},x))\\ &\overset{\widehat{2}}{=} A(A((a_1^{n-2},a)^{-1},a_1^{n-2},a),a_1^{n-2},x))\\ &\overset{\widehat{3}}{=} A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x).\end{aligned}$$

Finally, by $\widehat{2}$, $\widehat{3}[=(3_L)]$, $\widehat{4}[=(2_L)]$, and by 3.4 – $(a_1)[=Th.3.1]$, we conclude that Th. 3.4 – (a_5) holds.

$i=6$: For $n=2$ the case "$i=6$" coincides with the case "$i=5$". Let $n\geq 3$. Firstly we prove the following statements:

$\overline{\overline{1}}$ $A(x,a,a_1^{n-2})=A(y,a,a_1^{n-2})\Rightarrow x=y$ for all $x,y,a,a_1^{n-2}\in Q$;

$\overline{\overline{2}}$ (Q,A) is an $n-$semigroup;

$\overline{\overline{3}}$ $A((a_1^{n-2},a)^{-1},a_1^{n-2},a)=\mathbf{e}(a_1^{n-2})$ for all $a,a_1^{n-2}\in Q$; and

$\overline{\overline{4}}$ $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$ for all $x,a_1^{n-2}\in Q$.

Sketch of the proof of $\overline{\overline{1}}$:

$$\begin{aligned}&A(x,a,a_1^{n-2})=A(y,a,a_1^{n-2})\Rightarrow\\ &A(A(x,a,a_1^{n-2}),\mathbf{e}(a_1^{n-2}),\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))=A(A(y,a,a_1^{n-2}),\mathbf{e}(a_1^{n-2}),\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))\Rightarrow\\ &A(x,A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})),\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))=A(y,A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})),\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))\Rightarrow\\ &A(x,a,\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))=A(y,a,\overset{n-3}{a},\mathbf{e}(\overset{n-2}{a}))\Rightarrow x=y.\end{aligned}$$

The proof of $\overline{\overline{2}}$: By $\overline{\overline{1}}$, and by Prop. 2.1.

Sketch of the proof of $\overline{\overline{3}}$:

$$\begin{aligned}\mathbf{e}(a_1^{n-2}) &\overset{(4_L)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})))\\ &\overset{\overline{\overline{2}}}{=} A(A((a_1^{n-2},a)^{-1},a_1^{n-2},a),a_1^{n-2},\mathbf{e}(a_1^{n-2}))\\ &\overset{(2_R)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},a).\end{aligned}$$

Sketch of the proof of $\overline{\overline{4}}$:

$$\begin{aligned}x &\overset{(4_L)}{=} A((a_1^{n-2},a)^{-1},a_1^{n-2},A(a,a_1^{n-2},x))\\ &\overset{\overline{\overline{2}}}{=} A(A((a_1^{n-2},a)^{-1},a_1^{n-2},a),a_1^{n-2},x)\\ &\overset{\overline{\overline{3}}}{=} A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x).\end{aligned}$$

Finally, by $\overline{\overline{2}}$, $\overline{\overline{3}}[=(3_L)]$, $\overline{\overline{4}}[=(2_L)]$, and Th. 3.4 – (a_1) [=Th. 3.1], we conclude that Th. 3.4 – (a_6) holds.
3) The proof of (b_i), $i \in \{2, \dots, 6\}$:

By Th. 3.4$-(a_i)$, by Th. 2.6 from Chapter II, by Prop. 2.3 from Chapter II, and by Def. 1.1 from Chapter I. (See, the proof of Th. 3.2.)
4) The proof of (c_i), $i \in \{2, \dots, 6\}$:
4_1) The proof of (c_2) :

a) The laws (1_L) and (2_R) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_1$ for all $x_1^n \in Q$, $^{-1}$ an arbitrary $(n-1)-$ary operation in Q and $\mathbf{e}(a_1^{n-2}) = c$ for every sequence a_1^{n-2} over Q. However, the law (3_R) does not hold.

b) The laws (1_L) and (3_R) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_n$ for all $x_1^n \in Q$, $\mathbf{e}(a_1^{n-2}) = c$ for every sequence a_1^{n-2} over Q and $(a_1^{n-2}, a)^{-1} = \mathbf{e}(a_1^{n-2})$ for all $a, a_1^{n-2} \in Q$. However, the law (2_R) does not hold.

c) See the proof of Th. 3.3.

4_2) The proof of (c_3) :

a) The laws (1_L) and (2_L) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_n$ for all $x_1^n \in Q$, $\mathbf{e}$ an arbitrary $(n-2)-$ary operation in Q, and $(a_1^{n-2}, a)^{-1} = c$ for all $a, a_1^{n-2} \in Q$. However, the law (4_R) does not hold.

b) The laws (1_L) and (4_R) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) = x_1$ for all $x_1^n \in Q$, $^{-1}$ an arbitrary $(n-1)-$ary operation in Q, and $\mathbf{e}(a_1^{n-2}) = c$ for every sequence a_1^{n-2} over Q. However, the law (2_L) does not hold.

c) See the proof of Th. 3.3. (In Moufang's loop (Q, A), also the following laws hold: $A(a^{-1}, A(a, x)) = x$ and $A(A(x, a), a^{-1}) = x$; for example [Bruck 1958] and [Belousov 1967].)

Similarly, it is possible to prove the statements $(c_4) - (c_6)$. □

3.5. Remark: *Th. 3.4–(a_1) for $n = 2$ is proved in [Dickson 1905]. (Cf. [Cliford, Preston 1964].)*

Chapter IV

HOSSZÚ-GLUSKIN ALGEBRAS AND $n-$GROUPS

1 Auxiliary propositions

1.1. Proposition *[Ušan 1995/1]: Let (Q,A) be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$neutral operation and $n \geq 3$. Then, for every $a_1^{n-2}, b_1^{n-2}, x \in Q$ and for all $i \in \{1,\dots,n-1\}$ the following equalities hold*

$$A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) \text{ and}$$
$$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) = A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})).$$

Sketch of the proof.

1) $F(x, b_1^{n-2}) \overset{def}{=} A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Rightarrow$
$A(F(x,b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Rightarrow$
$A(F(x,b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(x, b_i^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{i-1}) \Rightarrow$
$A(F(x,b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Rightarrow$
$F(x, b_1^{n-2}) = x \Rightarrow$
$A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x);$

2) $F(x, b_1^{n-2}) \overset{def}{=} A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) \Rightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x,b_1^{n-2})) = A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x)) \Rightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x,b_1^{n-2})) = A(b_i^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{i-1}, x) \Rightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x,b_1^{n-2})) = A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) \Rightarrow$
$F(x, b_1^{n-2}) = x \Rightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) = A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})).$ □

1.2. Proposition*[Ušan 1995/1]: Let (Q,A) be an $n-$group, $^{-1}$ its inverse operation and $n \geq 2$. Then for every sequence a_1^{n-1} over Q there is exactly one $x \in Q$ such that the equality*

$$(a_1^{n-2}, x)^{-1} = a_{n-1}$$

holds.

Proof. Firstly we prove the following statements:

1° For all $a_1^{n-2}, x \in Q$ the following equality holds

(1) $$(a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1} = x; \text{ and}$$

2° For all $a_1^{n-2}, x, y \in Q$ the following equivalence holds

(2) $$(a_1^{n-2}, x)^{-1} = a_{n-1} \Leftrightarrow x = (a_1^{n-1})^{-1}.$$

The proof of the statement 1° :

By Th. 1.3 from Chapter III, for every sequence a_1^{n-2} over Q and for every $x \in Q$ the following equalities hold

$$A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, x) = \mathbf{e}(a_1^{n-2}) \text{ and}$$
$$A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, (a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1}) = \mathbf{e}(a_1^{n-2}).$$

Since (Q, A) is an $n-$quasigroup, we conclude that for every $a_1^{n-2}, x \in Q$ the following equality holds

$$(a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1} = x.$$

The proof of the statement 2° :

By 1°, we conclude that for all $a_1^{n-2}, x, y \in Q$ the following sequence of implications holds

$$(a_1^{n-2}, x)^{-1} = (a_1^{n-2}, y)^{-1} \Rightarrow$$
$$(a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1} = (a_1^{n-2}, (a_1^{n-2}, y)^{-1})^{-1} \Rightarrow x = y.$$

Hence, we conclude that also for all $a_1^{n-2}, x, y \in Q$ we have

$$(a_1^{n-2}, x)^{-1} = (a_1^{n-2}, y)^{-1} \Leftrightarrow x = y.$$

Finally, by 1° and 2°, we have that for every sequence a_1^{n-1} over Q and for every $x \in Q$ the following sequence of equivalences holds

$$(a_1^{n-2}, x)^{-1} = a_{n-1} \Leftrightarrow$$
$$(a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1} = (a_1^{n-1})^{-1} \Leftrightarrow x = (a_1^{n-1})^{-1},$$

and hence, we conclude that for every $a_1^{n-1} \in Q$ the following equivalence holds

$(a_1^{n-2}, x)^{-1} = a_{n-1} \Leftrightarrow x = (a_1^{n-1})^{-1}$.

Remark: For $n = 2$, $^{-1} \in Q$!.

1.3. Proposition *[Ušan 1994]: Let (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation, $^{-1}$ its inverse operation and $n \geq 2$. Then for every $a_1^{n-2}, b_1^{n-2}, x, y \in Q$ the following equality holds*

(a) $\quad A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1}), a_1^{n-2}, y)$.

Proof. 1) For $n = 2$ the equality (a) reduces to the equality:

$A(x, y) = A(x, y)$.

2) Let $n \geq 3$. Let also $x, y, a_1^{n-2}, b_1^{n-2}$ be arbitrary elements of the set Q. Then, by the assumptions, we have the following equalities

$$\begin{aligned} A(z, a_1^{n-2}, y) &= A(z, a_1^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, y)) \\ &= A(A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{n-2}, y), \end{aligned}$$

i.e. the equality

$A(A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{n-2}, y) = A(z, a_1^{n-2}, y)$.

Thereby, since for all $x, y, a_1^{n-2}, b_1^{n-2} \in Q$ the equivalence

$A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})) = x \Leftrightarrow z = A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1})$

holds, we conclude that the equality (a) is satisfied. □

1.4. Proposition *[Ušan 1995/1]: Let (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation and $n \geq 3$. Then for every sequence a_1^{n-2} over Q and for every $i \in \{1, \ldots, n-2\}$ there is exactly one $x_i \in Q$ such that the equality*

$$\mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2}$$

holds.

Proof. By Proposition 1.2 and Proposition 1.3, we conclude that for every $i \in \{1, \ldots, n-2\}$, for every sequence a_1^{n-2} over Q and for every $x_i \in Q$, the following sequence of equivalences holds

$\mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2} \overset{1.2}{\Leftrightarrow}$

$(a_1^{n-2}, \mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}))^{-1} = (a_1^{n-2}, a_{n-2})^{-1} \Leftrightarrow$

$$\begin{aligned}
&A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}))^{-1}) = \\
&A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a_{n-2})^{-1}) \Leftrightarrow \\
&A(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}))^{-1}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = \\
&A(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a_{n-2})^{-1}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) \Leftrightarrow^1 \\
&A(\mathbf{e}(a_1^{n-2}), a_1^{i-1}, x_i, a_i^{n-3}, \mathbf{e}(a_1^{n-2})) = \\
&A(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a_{n-2})^{-1}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) \Leftrightarrow \\
&A(\mathbf{e}(a_1^{n-2}), a_1^{i-1}, x_i, a_i^{n-3}, \mathbf{e}(a_1^{n-2})) = (a_1^{n-2}, a_{n-2})^{-1},
\end{aligned}$$

and hence we conclude that the equivalence

$$\begin{aligned}
&\mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2} \Leftrightarrow \\
&A(\mathbf{e}(a_1^{n-2}), a_1^{i-1}, x_i, a_i^{n-3}, \mathbf{e}(a_1^{n-2})) = (a_1^{n-2}, a_{n-2})^{-1}
\end{aligned}$$

holds for every $i \in \{1, \ldots, n-2\}$ and for every $a_1^{n-2}, x_i \in Q$. Hence, since (Q, A) is an $n-$quasigroup, we conclude that the proposition is true. □

2 Hosszú–Gluskin algebras

2.1. Definition *[Ušan 1995/1]: We say that an algebra* $(Q, \{\cdot, \varphi, b\})$ *[of the type* $< 2, 1, 0 >$*] is a* **Hosszú–Gluskin algebra of order** n $(n \geq 3)$ *[briefly: nHG−algebra] iff the following statements hold: (1)* $(Q, \cdot)$ *is a group; (2)* $\varphi \in Aut(Q, \cdot)$*; (3)* $\varphi(b) = b$*; and (4) for every* $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$.

See Example 1.4 from Chapter I.

2.2. Proposition: *Let* $(Q, \{\cdot, \varphi, b\})$ *be an nHG−algebra. Also, let*
(5) $A(x_1^n) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$
for all $x_1^n \in Q$*. Then* (Q, A) *is an* $n-$*group.*

Proof. Firstly we prove the following statements:

1° (Q, A) is an $n-$quasigroup; and

2° (Q, A) is an $< 1, 2 > -$associative $n-$groupoid.

The proof of the statement 1° :

By (1),(2) and (5).

[1]Proposition 1.3: $x = y = \mathbf{e}(a_1^{n-2})$ and $b_1^{n-2} = a_1^{i-1}, x_i, a_i^{n-3}$.

Sketch of the proof of 2° :
$A(A(x_1^n), x_{n+1}^{2n-1}) =$
$(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b) \cdot \varphi(x_{n+1}) \cdot \varphi^2(x_{n+2}) \cdot \ldots \cdot \varphi^{n-1}(x_{2n-1}) \cdot b =$
$x_1 \cdot (\varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \cdot \varphi(x_{n+1})) \cdot \varphi^2(x_{n+2}) \cdot \ldots \cdot \varphi^{n-1}(x_{2n-1}) \cdot b =$
$x_1 \cdot (\varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot \varphi(b) \cdot \varphi(x_{n+1})) \cdot \varphi^2(x_{n+2}) \cdot \ldots \cdot \varphi^{n-1}(x_{2n-1}) \cdot b =$
$x_1 \cdot \varphi(x_2 \cdot \ldots \cdot \varphi^{n-2}(x_n) \cdot b \cdot x_{n+1}) \cdot \varphi^2(x_{n+2}) \cdot \ldots \cdot \varphi^{n-1}(x_{2n-1}) \cdot b =$
$x_1 \cdot \varphi(x_2 \cdot \ldots \cdot \varphi^{n-2}(x_n) \cdot \varphi^{n-1}(x_{n+1}) \cdot b) \cdot \varphi^2(x_{n+2}) \cdot \ldots \cdot \varphi^{n-1}(x_{2n-1}) \cdot b =$
$A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$.

Finally, by 1°, 2° and Proposition 2.3 from Chapter III, we conclude that the proposition is satisfied. □

2.3. Definition *[Ušan 1995/1]: We say that an $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ is* **associated** *(or corresponds) to the $n-$group (Q, A) iff the equality (5) holds for all $x_1^n \in Q$.*

3 A proof of the Hosszú – Gluskin Theorem

3.1. Theorem *(Hosszú – Gluskin Theorem): Let (Q, A) be an $n-$group,* $\mathbf{e}$ *its $\{1, n\}-$neutral operation and $n \geq 3$. Let also c_1^{n-2} be an arbitrary sequence over Q, and let*

(a) $x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y)$,

(b) $\varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$ *and*

(c) $b \overset{def}{=} A(\overline{\mathbf{e}(c_1^{n-2})|}^{n})$

for all $x, y \in Q$. Then, the following statements hold: (i) $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra; and (ii) for every $x_1^n \in Q$ the equality

(d) $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$

holds.[2]

[2][Ušan 1995/1]. The proof of the theorem follow the **idea of E.I. Sokolov** from [Sokolov 1976]. In detail in Appendix 2.

See, also Ch.I-2.1.

Proof.[3] Firstly we prove that the stetements (1)-(4) from Def. 2.1 hold.

The proof of the statement (1): By (Q, A) is an $n-$group.

Sketch of the proof of (2):

$$\begin{aligned}\varphi(x\cdot y) &\overset{(a),(b)}{=} A(\mathbf{e}(c_1^{n-2}), A(x, c_1^{n-2}, y), c_1^{n-2})\\ &= A(A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}), y, c_1^{n-2}) \overset{(b)}{=} A(\varphi(x), y, c_1^{n-2})\\ &\overset{1.1}{=} A(\varphi(x), A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), y), c_1^{n-2})\\ &= A(\varphi(x), c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), y, c_1^{n-2})) \overset{(b)}{=} A(\varphi(x), c_1^{n-2}, \varphi(y))\\ &\overset{(a)}{=} \varphi(x)\cdot\varphi(y).\end{aligned}$$

Sketch of the proof of (3):

$$\begin{aligned}\varphi(b) &\overset{(c)}{=} \varphi(A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|)) \overset{(b)}{=} A(\mathbf{e}(c_1^{n-2}), A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|), c_1^{n-2})\\ &= A(A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) \overset{1.1}{=} A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|) \overset{(c)}{=} b.\end{aligned}$$

Sketch of the proof of (4):

$$\begin{aligned}b\cdot x &\overset{(a),(c)}{=} A(A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|), c_1^{n-2}, x)\\ &= A(\overset{n-1}{\overline{\mathbf{e}(c_1^{n-2})}}|, A(\mathbf{e}(c_1^{n-2}), c_1^{n-2}, x))\\ &= A(\overset{n-1}{\overline{\mathbf{e}(c_1^{n-2})}}|, A(x, c_1^{n-2}, \mathbf{e}(c_1^{n-2})))\\ &= A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}|, A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}), \mathbf{e}(c_1^{n-2}))\\ &\overset{(b)}{=} A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}|, \varphi(x), \mathbf{e}(c_1^{n-2}))\\ &= A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}|, A(\varphi(x), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2}))\\ &= A(\overset{n-3}{\overline{\mathbf{e}(c_1^{n-2})}}|, A(\mathbf{e}(c_1^{n-2}), \varphi(x), c_1^{n-2}), \overset{2}{\overline{\mathbf{e}(c_1^{n-2})}}|)\end{aligned}$$

---

[3] [Ušan 1995/1].

$= A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n-3}|, \varphi^2(x), \overbrace{\mathbf{e}(c_1^{n-2})}^{2}|)$

$- -$

$- -$

$= A(\varphi^{n-1}(x), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}|)$

$= A(A(\varphi^{n-1}(x), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}|)$

$= A(\varphi^{n-1}(x), c_1^{n-2}, A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}|)$

$\overset{(a),(c)}{=\!=} \varphi^{n-1}(x) \cdot b.$

Finally, we prove that the statement (d) holds.

By definition $(a)-(c)$, using the asumption that (Q, A) is an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation and $n \geq 3$, and by Proposition 1.1 from Chapter IV, we conclude that for every $x_1^n \in Q$ the following sequence of equalities holds:

$A(x_1^n) = A(x_1^{n-1}, A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), A(x_n, c_1^{n-2}, \mathbf{e}(c_1^{n-2})))) =$

$= A(x_1^{n-1}, A(c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), x_n, c_1^{n-2}), \mathbf{e}(c_1^{n-2}))) =$

$= A(x_1^{n-1}, A(c_1^{n-2}, \varphi(x_n), \mathbf{e}(c_1^{n-2}))) =$

$= A(x_1^{n-2}, A(x_{n-1}, c_1^{n-2}, \varphi(x_n)), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-2}, x_{n-1} \cdot \varphi(x_n), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-2}, A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), A(x_{n-1} \cdot \varphi(x_n), c_1^{n-2}, \mathbf{e}(c_1^{n-2}))), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-2}, A(c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), x_{n-1} \cdot \varphi(x_n), c_1^{n-2}), \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-2}, A(c_1^{n-2}, \varphi(x_{n-1} \cdot \varphi(x_n)), \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-3}, A(x_{n-2}, c_1^{n-2}, \varphi(x_{n-1} \cdot \varphi(x_n))), \mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2})) =$

$= A(x_1^{n-3}, x_{n-2} \cdot \varphi(x_{n-1} \cdot \varphi(x_n)), \mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2})) =$

$- -$

$- -$

$= A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}|)$

$= A(A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}|)$

$= A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), c_1^{n-2}, A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}|))$

$= x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$ □

4 Two descriptions of all $nHG-$algebras corresponding to the same $n-$group

4.1. Theorem *[Ušan 1995/1]: Let (Q,A) be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$neutral operation, and $n \geq 3$. Further on, let c_1^{n-2} be an arbitrary sequence over Q, and let for every $x,y \in Q$*

$$B_{(c_1^{n-2})}(x,y) \overset{def}{=} A(x, c_1^{n-2}, y),$$

$$\varphi_{(c_1^{n-2})}(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \text{ and}$$

$$b_{(c_1^{n-2})} \overset{def}{=} A(\overline{\overset{n}{\mathbf{e}(c_1^{n-2})}}|).$$

Let also

$$\mathcal{C}_A \overset{def}{=} \{(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) | c_1^{n-2} \in Q\}.$$

Then for every $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ the following equivalence holds

$$(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A \Leftrightarrow (\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

Proof. 1) $\Rightarrow$: By Theorem 3.1, we conclude that for every $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ the following implication holds:

$$(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A \Rightarrow (\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

2) $\Leftarrow$:

2_1 Let (Q,A) be an $n-$group, $n \geq 3$, $(Q, \{\cdot, \varphi, b\})$ an $nHG-$algebra, e the neutral element of the group $(Q, \cdot)$ and $^{\mathbf{-1}}$ the inverse operation in $(Q, \cdot)$. Let also for every $x_1^n \in Q$ the following equality holds:

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

If in the above equality we put $x_2^{n-2} = \overset{n-3}{e}$ and $x_{n-1} = b^{\mathbf{-1}}$, since $\varphi(b) = b$ and thus also $\varphi(b^{\mathbf{-1}}) = b^{\mathbf{-1}}$, we conclude that for every $x_1, x_n \in Q$ the following equality holds:

$$A(x_1, \overset{n-3}{e}, b^{\mathbf{-1}}, x_n) = x_1 \cdot x_n,$$

and hence we conclude that for all $x, y \in Q$ the equality

$$x \cdot y = B_{(\overset{n-3}{e}, b^{\mathbf{-1}})}(x,y)$$

also holds.

2_2 Let $(Q, \{\cdot, \varphi, b\})$ and $(Q, \{\cdot, \overline{\varphi}, \overline{b}\})$ be two $nHG-$ algebras, e the neutral element of the group $(Q, \cdot)$. Let also for every $a_1^n \in Q$ the following equality holds

$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n = x_1 \cdot \overline{\varphi}(x_2) \cdot \ldots \cdot \overline{\varphi}^{n-2}(x_{n-1}) \cdot \overline{b} \cdot x_n.$

If in the above equality we put $x_1 = \ldots = x_n = e$, we conclude that

$$b = \overline{b},$$

which means that for every $x_1^n \in Q$ the following equality holds

$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n = x_1 \cdot \overline{\varphi}(x_2) \cdot \ldots \cdot \overline{\varphi}^{n-2}(x_{n-1}) \cdot b \cdot x_n,$

and hence, by similar argument, we conclude that

$$\varphi = \overline{\varphi}.$$

2_3 By Theorem 3.1 – (i) and by that the arguments from 2_1) and 2_2), we conclude that for every $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ the following implication holds

$(\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \Rightarrow (Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A.$ □

A consequence of Theorem 4.1 and Proposition 1.3 is the following proposition:

4.2. Proposition *[Ušan 1995/1]: Let (Q, A) be an $n-$group,* $\mathbf{e}$ *its $\{1, n\}-$neutral operation, $^{-1}$ its inverse operation and $n \geq 3$. Let also $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra corresponding to the $n-$group (Q, A), e the neutral element of the group $(Q, \cdot)$ and $^{\mathbf{-1}}$ the inverse operation in $(Q, \cdot)$. Then for every $b_1^{n-2} \in Q$ the following equality holds:*

$$\mathbf{e}(b_1^{n-2}) = (\varphi(b_1) \cdot \ldots \cdot \varphi^{n-2}(b_{n-2}) \cdot b)^{\mathbf{-1}}.$$

Proof. By Theorem 4.1, there is a sequence a_1^{n-2} over Q such that for all $x, y \in Q$ the equality

(a) $\quad x \cdot y = A(x, a_1^{n-2}, y)$

holds. The following also hold

(b) $\quad e = \mathbf{e}(a_1^{n-2}),$

and

(c) $\quad (\forall a \in Q) a^{\mathbf{-1}} = (a_1^{n-2}, a)^{-1}.$

Let also b_1^{n-2} be an arbitrary sequence over Q. Then, by Proposition 1.3, for all $x, y \in Q$ the following equality holds

$$A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1}), a_1^{n-2}, y),$$

i.e., by (a) and (c), also the equality

$$A(x, b_1^{n-2}, y) = x \cdot (\mathbf{e}(b_1^{n-2}))^{\mathbf{-1}} \cdot y.$$

Hence, since $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra corresponding to the $n-$group (Q, A), we conclude that for all $x, y \in Q$ the following holds

$$x \cdot \varphi(b_1) \cdot \ldots \cdot \varphi^{n-2}(b_{n-2}) \cdot b \cdot y = x \cdot (\mathbf{e}(b_1^{n-2}))^{\mathbf{-1}} \cdot y.$$

Hence, we conclude that the proposition holds. □

4.3. Theorem: *[Ušan 1995/1]: Let (Q, A) be an $n-$group, $n \geq 3$, $(Q, \{\cdot, \varphi, b\})$ an arbitrary $nHG-$algebra corresponding to the $n-$group (Q, A), $^{\mathbf{-1}}$ the inverse operation in $(Q, \cdot)$, $k \in Q$ and for every $x, y \in Q$*

(α) $\quad x \cdot_k y \overset{def}{=} x \cdot k \cdot y,$

(β) $\quad \varphi_k(x) \overset{def}{=} k^{\mathbf{-1}} \cdot \varphi(x) \cdot \varphi(k)$ *and*

(γ) $\quad b_k \overset{def}{=} k^{\mathbf{-1}} \cdot \varphi(k^{\mathbf{-1}}) \cdot \ldots \cdot \varphi^{n-1}(k^{\mathbf{-1}}) \cdot b.$

Let also

(δ) $\quad \widehat{\mathcal{C}}_A \overset{def}{=} \{(Q, \{\cdot_k, \varphi_k, b_k\}) | k \in Q\}.$

Then, $\widehat{\mathcal{C}}_A$ is a set of all $nHG-$algebras corresponding to the $n-$group (Q, A), i.e., then $\widehat{\mathcal{C}}_A = \mathcal{C}_A$.

Proof. 1) Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra corresponding to the $n-$group (Q, A). By Theorem 4.1,

$$(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A,$$

i.e., there is a sequence a_1^{n-2} over Q such that

$$x \cdot y = A(x, a_1^{n-2}, y),$$
$$\varphi(x) = A(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2}) \text{ and}$$
$$b = A(\overset{n}{\overline{\mathbf{e}(a_1^{n-2})}}|).$$

In addition, by $(\alpha) - (\beta)$, we conclude that $\cdot = \cdot_e$, $\varphi = \varphi_e$ and $b = b_e$, where e is the neutral element of the group $(Q, \cdot)$, and hence also

$$(Q, \{\cdot, \varphi, b\}) \in \widehat{\mathcal{C}}_A.$$

Thus,

$$(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A \cap \widehat{\mathcal{C}}_A$$

also holds.

2) Let also $(Q, \{\diamond, \varphi_\diamond, b_\diamond\})$ be an arbitrary $nHG-$algebra corresponding to the $n-$group (Q, A). By Theorem 4.1,

$$(Q, \{\diamond, \varphi_\diamond, b_\diamond\}) \in \mathcal{C}_A,$$

i.e. there is a sequence b_1^{n-2} over Q such that

$$x \diamond y = A(x, b_1^{n-2}, y),$$
$$\varphi_\diamond = A(\mathbf{e}(b_1^{n-2}), x, b_1^{n-2}) \text{ and}$$
$$b_\diamond = A(\overset{n}{\overline{\mathbf{e}(b_1^{n-2})}}|).$$

By Proposition 1.3, for all $x, y \in Q$ the equality

$$A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1}), a_1^{n-2}, y)$$

holds. Since a_1^{n-2}, b_1^{n-2} are fixed elements of the set Q, there is $k \in Q$ such that

$$(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1} = k.$$

Hence, for all $x, y \in Q$, the equality

$$x \diamond y = x \cdot k \cdot y$$

holds. In addition

$$\mathbf{e}(b_1^{n-2}) = k^{\mathbf{-1}};$$

for the inverse operation $^{\mathbf{-1}}$ in the group $(Q, \cdot)$, namely, the following holds

$$a^{\mathbf{-1}} = (a_1^{n-2}, a)^{-1}$$

for every $a \in Q$. Thereby, and by the assumption that $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra corresponding to the $n-$group (Q, A), by Proposition 4.2, we conclude that for every $x \in Q$ the following sequence of equalities holds

$$\begin{aligned}
\varphi_\diamond(x) &= A(\mathbf{e}(b_1^{n-2}), x, b_1^{n-2}) \\
&= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot \ldots \cdot \varphi^{n-1}(b_{n-2}) \cdot b \\
&= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot \ldots \cdot \varphi^{n-1}(b_{n-2}) \cdot \varphi(b) \\
&= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi(\varphi(b_1) \cdot \ldots \cdot \varphi^{n-2}(b_{n-2}) \cdot b) \\
&= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi((\mathbf{e}(b_1^{n-2}))^{\mathbf{-1}}) \\
&= k^{\mathbf{-1}} \cdot \varphi(x) \cdot \varphi(k),
\end{aligned}$$

hence we conclude that for every $x \in Q$ the equality

$$\varphi_\diamond(x) = k^{\mathbf{-1}} \cdot \varphi(x) \cdot \varphi(k)$$

holds. Similarly, we conclude that also the following sequence of equalities holds

$$\begin{aligned} b_\diamond &= A(\overset{n}{\overline{\mathbf{e}(b_1^{n-2})}}|) \\ &= \mathbf{e}(b_1^{n-2}) \cdot \varphi(\mathbf{e}(b_1^{n-2})) \cdot \ldots \cdot \varphi^{n-1}(\mathbf{e}(b_1^{n-2})) \cdot b \\ &= k^{\mathbf{-1}} \cdot \varphi(k^{\mathbf{-1}}) \cdot \ldots \cdot \varphi^{n-1}(k^{\mathbf{-1}}) \cdot b, \end{aligned}$$

and hence we conclude

$$b_\diamond = k^{\mathbf{-1}} \cdot \varphi(k^{\mathbf{-1}}) \cdot \ldots \cdot \varphi^{n-1}(k^{\mathbf{-1}}) \cdot b.$$

Thus,

$$\mathcal{C}_A \subseteq \widehat{\mathcal{C}}_A.$$

holds.

3) Finally, let $(Q, \{\diamond, \varphi_\diamond, b_\diamond)\})$ be an arbitrary element from the set $\widehat{\mathcal{C}}_A$. Then, by $(\alpha) - (\delta)$, there is $k \in Q$ such that for all $x, y \in Q$ the equalities

$$x \diamond y = x \cdot k \cdot y,$$
$$\varphi_\diamond(x) = k^{\mathbf{-1}} \cdot \varphi(x) \cdot \varphi(k) \text{ and}$$
$$b_\diamond = k^{\mathbf{-1}} \cdot \varphi(k^{\mathbf{-1}}) \cdot \ldots \cdot \varphi^{n-1}(k^{\mathbf{-1}}) \cdot b$$

hold. In addition, by Proposition 1.2 and Proposition 1.4, we conclude that for every sequence b_1^{n-3} over Q and for every $i \in \{1, \ldots, n-2\}$, there is exactly one $x_i \in Q$ such that

$$(a_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))^{-1} = k.$$

Hence, firstly, by Proposition 1.3, we conclude that for all $x, y \in Q$ the following sequence of equalities holds

$$\begin{aligned} x \diamond y &= x \cdot k \cdot y \\ &= A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))^{-1}), a_1^{n-2}, y) \\ &= A(x, b_1^{i-1}, x_i, b_i^{n-3}, y), \end{aligned}$$

i.e. that for all $x, y \in Q$ also the equality

$$x \diamond y = A(x, b_1^{i-1}, x_i, b_i^{n-3}, y)$$

holds. Further, by the similar argument as in 2), we conclude that for every $x \in Q$ the following sequence of equalities holds

$$\begin{aligned} \varphi_\diamond(x) &= k^{\mathbf{-1}} \cdot \varphi(x) \cdot \varphi(k) \\ &= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))^{\mathbf{-1}}) \\ &\overset{4.2}{=} \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi(\varphi(b_1) \cdot \ldots \cdot \varphi^i(x_i) \cdot \ldots \cdot \varphi^{n-2}(b_{n-3}) \cdot b) \end{aligned}$$

$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot \ldots \cdot \varphi^{i+1}(x_i) \cdot \ldots \cdot \varphi^{n-1}(b_{n-3}) \cdot \varphi(b)$
$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot \ldots \cdot \varphi^{i+1}(x_i) \cdot \ldots \cdot \varphi^{n-1}(b_{n-3}) \cdot b$
$= A(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}), x, b_i^{i-1}, x_i, b_i^{n-3}),$
i.e. that for every $x \in Q$ the equality

$$\varphi_\diamond(x) = A(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}), x, b_1^{i-1}, x_i, b_i^{n-3})$$

is satisfied. Finally, since $k^{-\mathbf{1}} = \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})$ and by the assumption that $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra corresponding to the $n-$group (Q, A), we conclude that

$$\begin{aligned} b_\diamond &= k^{-\mathbf{1}} \cdot \varphi(k^{-\mathbf{1}}) \cdot \ldots \cdot \varphi^{n-1}(k^{-\mathbf{1}}) \cdot b \\ &= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \ldots \cdot \varphi^{n-1}(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})) \cdot b \\ &= A(\overline{\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})}^{\,n}|); \end{aligned}$$

i.e. that

$$b_\diamond = A(\overline{\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})}^{\,n}|),$$

and hence

$$\widehat{\mathcal{C}}_A \subseteq \mathcal{C}_A$$

also holds. □

4.4. Remark*: One generalization of Hosszú-Gluskin Theorem is given in [Dudek, Michalski 1982].*

Chapter V

$n-$GROUPS WITH $\{i,j\}-$NEUTRAL OPERATIONS FOR $\{i,j\}\neq\{1,n\}$

1 Main proposition

1.1. Theorem *[Ušan 1995/2]: Let (Q,A) be an $n-$group, $n\geq 3$, $(i,j)\in\{1,\dots,n\}^2, i<j$ and $\{i,j\}\neq\{1,n\}$. Let also $(Q,\{\cdot,\varphi,b\})$ be an $nHG-$algebra associated to the $n-$group (Q,A). Then, the following statements are equivalent:*

(i) (Q,A) has an $\{i,j\}-$neutral operation; and

(ii) $(Q,\cdot)$ is a commutative group, $\varphi^{i-1}=I$ and $\varphi^{j-1}=I$, where I is the identity permutation of the set Q.

Proof. 1) One convention: Let $(Q,\cdot)$ be a group (semigroup) with the neutral element e. In the proof we use the following convention:

$$(1)\qquad \prod_{t=p}^{q} c_t \text{ stands for } \begin{cases} c_p\cdot\ldots\cdot c_q & ,p<q\\ c_p & ,p=q\\ e & ,c_p^q=\emptyset.\end{cases}\quad [1]$$

For example

$$\prod_{t=n+1}^{n}\varphi^{t-1}(b_{t-2})=e\ (:\ \overline{\varphi^{t-1}(b_{t-2})}|_{t=n+1}^{n}=\emptyset).$$

2) $(i)\Rightarrow(ii)$: Let (Q,A) be an $n-$group, $n\geq 3$ and let $(Q,\{\cdot,\varphi,b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q,A). Also let e the neutral element of the group $(Q,\cdot)$ and $^{-1}$ the inverse operation in $(Q,\cdot)$. Further on, let E be an $\{i,j\}-$neutral operation of (Q,A) such that the

[1] The case $p\leq q$ is based on the following convention: $\prod_{t=1}^{n+1} c_t \overset{def}{=} (\prod_{t=1}^{n} c_t)\cdot c_{n+1}$ and $\prod_{t=m}^{m} c_t \overset{def}{=} c_m$.

condition

(2) $\{i,j\} \neq \{1,n\}$

holds. Then, using Th. 3.1 from Chapter IV and convention (1),we conclude that for every $x, b_1^{n-2} \in Q$ the following equalities hold:

(3) $(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \varphi^{i-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=i}^{j-2} \varphi^t(b_t)) \cdot \varphi^{j-1}(x) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x$

and

(4) $(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \varphi^{i-1}(x) \cdot (\prod_{t=i}^{j-2} \varphi^t(b_t)) \cdot \varphi^{j-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x,$

hence, we have that for every $x, b_1^{n-2} \in Q$ the following equality holds

(5) $\varphi^{i-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=i}^{j-2} \varphi^t(b_t)) \cdot \varphi^{j-1}(x) = \varphi^{i-1}(x) \cdot (\prod_{t=i}^{j-2} \varphi^t(b_t)) \cdot \varphi^{j-1}(\mathsf{E}(b_1^{n-2})).$

Substituing x by e, we deduce from (3) that for every sequence b_1^{n-2} over Q the following equality holds

(6) $\varphi^{i-1}(\mathsf{E}(b_1^{n-2})) = (\prod_{t=1}^{i-1} \varphi^{t-1}(b_t))^{-1} \cdot b^{-1} \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2}))^{-1} \cdot (\prod_{t=i}^{j-2} \varphi^t(b_t))^{-1},$

hence, since $(Q,\cdot)$ is a group and φ and $^{-1}$ are permutations of the set Q, we conclude that the following holds:

1° E is a permutation of the set Q for $n = 3$, and (Q, E) is an $(n-2)-$quasigroup for $n \geq 4$.

The consequence of the condition (2) is the following:

2° If $\overline{\varphi^t(b_t)}|_{t=i}^{j-2}$ is not empty sequence, then at least one of the variables b_1 and b_{n-2} is not the variable in the term $\prod_{t=i}^{j-2} \varphi^t(b_t)$.

Since $\varphi(b^{-1}) = b^{-1}$ [:Def. 2.1 from Chapter IV and $b \cdot b^{-1} = e$], if we put in (6) $b_1 = \ldots = b_{n-2} = e$, we conclude that the following statement holds:

3° $\mathsf{E}(\overset{n-2}{e}) = b^{-1}$.

By 3°, putting in (4) $b_1 = \ldots = b_{n-2} = e$, we conclude that the following statement holds:

4° $\varphi^{i-1} = I$.

By 3°, putting in (3) $b_1 = \ldots = b_{n-2} = e$, and by Def. 2.1 from Chapter

IV, we conclude that the following statement holds

4°_1 $\varphi^{j-1}=\varphi^{n-1}$.

Sketch of the proof of 4°_1 :
$\varphi^{i-1}(b^{-1})\cdot\varphi^{j-1}(x)\cdot b=x\Rightarrow b^{-1}\cdot\varphi^{j-1}(x)\cdot b=x\Rightarrow$
$\varphi^{j-1}(x)\cdot b=b\cdot x\Rightarrow\varphi^{j-1}(x)\cdot b=\varphi^{n-1}(x)\cdot b\Rightarrow$
$\varphi^{j-1}(x)=\varphi^{n-1}(x)$.

By $1^\circ,\ 2^\circ,\ 4^\circ,\ 4^\circ_1$, by statement connected with (5) and by Def. 2.1 from Chapter IV, we conclude that also the following statement holds:

5° $(Q,\cdot)$ is a commutative group.

Sketch of the proof of 5° :
Let $\mathsf{E}(b_1,\overset{n-3}{e})=y$ (if $b_i\neq b_1$) or $\mathsf{E}(\overset{n-3}{e},b_{n-2})=y$ (if $b_{j-2}\neq b_{n-2}$) [:$1^\circ,\ 2^\circ$]. Then:
$\varphi^{i-1}(y)\cdot\varphi^{j-1}(x)=\varphi^{i-1}(x)\cdot\varphi^{j-1}(y)\overset{4^\circ}{\Rightarrow}y\cdot\varphi^{j-1}(x)=x\cdot\varphi^{j-1}(y)\overset{4^\circ_1}{\Rightarrow}$
$y\cdot\varphi^{n-1}(x)=x\cdot\varphi^{n-1}(y)\Rightarrow y\cdot\varphi^{n-1}(x)\cdot b=x\cdot\varphi^{n-1}(y)\cdot b\Rightarrow$
$y\cdot b\cdot x=x\cdot b\cdot y\Rightarrow(by)\cdot(bx)=(bx)\cdot(by)$.
In addition, by $5^\circ,\ 3^\circ$, putting in (3) $b_1=\ldots=b_{n-2}=e$, we conclude

6° $\varphi^{j-1}=I$.

By $4^\circ,\ 5^\circ$, and 6°, we finally conclude that the implication $(i)\Rightarrow(ii)$ holds.

3) $(ii)\Rightarrow(i)$: Let (Q,A) be an $n-$group, $n\geq 3$, let $(Q,\{\cdot,\varphi,b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q,A), and let the statement (ii) holds. Let also, for every $b_1^{n-2}\in Q$
$\mathsf{E}(b_1^{n-2})\overset{def}{=}(\prod\limits_{t=1}^{i-1}\varphi^{t-1}(b_t))^{-1}\cdot b^{-1}\cdot(\prod\limits_{t=j+1}^{n}\varphi^{t-1}(b_{t-2}))^{-1}\cdot(\prod\limits_{t=i}^{j-2}\varphi^{t}(b_t))^{-1}$.
Then, for every $x,b_1^{n-2}\in Q$, the following two sequences of equalities hold:
$A(b_1^{i-1},\mathsf{E}(b_1^{n-2}),b_i^{j-2},x,b_{j-1}^{n-2})=$
$(\prod\limits_{t=1}^{i-1}\varphi^{t-1}(b_t))\cdot\mathsf{E}(b_1^{n-2})\cdot(\prod\limits_{t=i}^{j-2}\varphi^{t}(b_t))\cdot x\cdot(\prod\limits_{t=j+1}^{n}\varphi^{t-1}(b_{t-2}))\cdot b=x$, and
$A(b_1^{i-1},x,b_i^{j-2},\mathsf{E}(b_1^{n-2}),b_{j-1}^{n-2})=$

$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot x \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \mathsf{E}(b_1^{n-2}) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x.$$

Hence, the implication $(ii) \Rightarrow (i)$ also holds. $\square$

1.2. Remark: *If $(Q,\cdot)$ is a noncommutative group and $A(x_1^n) = x_1 \cdot \ldots \cdot x_n$, $n \geq 3$, then (Q,A) is an $n-$group without $\{i,j\}-$neutral operations with condition $\{i,j\} \neq \{1,n\}$ (:Theorem 1.1; $\neg(i) \Leftrightarrow \neg(ii)$). Besides, for example, if $(Q,\cdot)$ is a commutative group in which not every $a \in Q$ is selfinverse, $^{-1}$ is the inverse operation in $(Q,\cdot)$ and $A(x_1^3) \overset{def}{=} x_1 \cdot x_2^{-1} \cdot x_3$, then (Q,A) is a 3-group without $\{i,j\}-$neutral operations with the condition $\{i,j\} \neq \{1,n\}$ (:Theorem 1.1; $\varphi =^{-1}$).*

2 Two propositions more

2.1. Theorem *[Ušan 1995/2]: Let $n \geq 3$, (Q,A) be an $n-$group, and* $\mathbf{e}$ *its $\{1,n\}-$neutral operation [Th. 2.6 from Chapter II]. Then the following statements are equaivalent:*

(i) (Q,A) is a commutative $n-$group;

(ii) $\mathbf{e}$ *is an $\{i,j\}-$neutral operation of the $n-$group (Q,A) for every $(i,j) \in \{(p,q)|(p,q) \in \{1,\ldots,n\}^2 \wedge p < q\}$;*

(iii) (Q,A) has an $\{1,n-1\}-$neutral operation; and

(iv) (Q,A) has a $\{2,n\}-$neutral operation.

Proof. 1) $(i) \Rightarrow (ii)$: (Q,A) is commutative iff for every permutation α of the set $\{1,\ldots,n\}$ and for every $x_1^n \in Q$ the following equality holds

$A(x_{\alpha(1)},\ldots,x_{\alpha(n)}) = A(x_1^n)$.

Hence, by Th. 2.6 from Chapter II, we conclude that the implication $(i) \Rightarrow (ii)$ holds.

2) $(ii) \Rightarrow (iii)$: By (ii), $\mathbf{e}$ is also an $\{1,n-1\}-$neutral operation of the $n-$group (Q,A).

3) $(iii) \Rightarrow (iv)$: Let $(Q,\{\cdot,\varphi,b\})$ be an $nHG-$algebra associated to the $n-$group (Q,A). Hence, by Th. 1.1 and by the condition (iii), we conclude

that:

a) $(Q,\cdot)$ is a commutative group; and

b) $\varphi^{n-2} = I$.

Further, by a) and by Def. 2.1-(4) from Chapter IV, we conclude that the following equality holds:

c) $\varphi^{n-1} = I$.

From b) and c) it follows that:

d) $\varphi = I$.

Finally, by a), c) and d), and by Th. 1.1, we conclude that (Q,A) has a $\{2,n\}-$neutral operation.

4) $(iv) \Rightarrow (i)$: Let $(Q,\{\cdot,\varphi,b\})$ be an $nHG-$algebra associated to the $n-$group (Q,A). Thereby, and also by Th. 1.1 and condition (iv), we conclude that $(Q,\cdot)$ is a commutative group and that $\varphi = I$, hence, by 2.3 from Chapter IV, we finally conclude that (Q,A) is a commutative $n-$group. □

A consequence of Th. 1.1 and of Th. 2.1 is the following proposition:

2.2. Corollary *[Ušan 1995/2]: Let (Q,A) be an $n-$group and $n \in \{3,4\}$. Then the following stetements are equaivalent:*

(a) (Q,A) has an $\{i,j\}-$neutral operation with the condition $\{i,j\} \neq \{1,n\}$; and

(b) (Q,A) is a commutative $n-$group. □

3 Example

Let $(Q,\cdot)$ be a commutative group in which not every $a \in Q$ is selfinverse, e its neutral element and $^{-1}$ its the inverse operation. Let also

$$A(x_1^5) \overset{def}{=} x_1 \cdot x_2^{-1} \cdot x_3 \cdot x_4^{-1} \cdot x_5.$$

Then:

a) (Q,A) is a 5-group (Prop. 2.2 from IV; $\varphi =^{-1},\ b = e$);

b) for

$$\mathbf{e}(a_1^3)\overset{def}{=}a_1\cdot a_2^{-1}\cdot a_3,$$

$\mathbf{e}$ is an $\{1,5\}-$neutral operation of the 5-group (Q,A);

c) for

$$\mathsf{E}_1(a_1^3)\overset{def}{=}a_1^{-1}\cdot a_2\cdot a_3,$$

E_1 is a $\{3,5\}-$neutral operation of the 5-group (Q,A); and

d) for

$$\mathsf{E}_2(a_1^3)\overset{def}{=}a_1\cdot a_2\cdot a_3^{-1},$$

E_2 is a $\{1,3\}-$neutral operation of the 5-group (Q,A). $\square$

Chapter VI

CONGRUENCES

1 On congruences in an $m-$groupoid

Let (Q, A) be an $m-$groupoid and $m \geq 1$. Let also θ be an **equivalence relation** in the set Q. Then θ is a **congruence relation** on the $m-$groupoid (Q, A) iff for each $a_1^m, b_1^m \in Q$ the following formula holds

(a) $\bigwedge\limits_{i=1}^{m} a_i \;\theta\; b_i \Rightarrow A(a_1^m) \;\theta\; A(b_1^m).$

The following **proposition** is true: θ is a **congruence** on an $m-$groupoid (Q, A) iff for each $a, b, c_1^{m-1} \in Q$ the following formula holds

(b) $\bigwedge\limits_{i=1}^{m} (a \;\theta\; b \Rightarrow A(c_1^{i-1}, a, c_i^{m-1}) \;\theta\; A(c_1^{i-1}, b, c_i^{m-1})).$

A **congruence relation** θ on an $m-$groupoid (Q, A) is said to be **normal** iff for each $a, b, c_1^{m-1} \in Q$ the following formula holds

(c) $\bigwedge\limits_{i=1}^{m} (A(c_1^{i-1}, a, c_i^{m-1}) \;\theta\; A(c_1^{i-1}, b, c_i^{m-1}) \Rightarrow a \;\theta\; b).$

Thus, an equivalence relation θ in a set Q is a **normal congruence relation** on an $m-$groupoid (Q, A) iff for each $a, b, c_1^{m-1} \in Q$ the following formula holds

(d) $\bigwedge\limits_{i=1}^{m} (a \;\theta\; b \Leftrightarrow A(c_1^{i-1}, a, c_i^{m-1}) \;\theta\; A(c_1^{i-1}, b, c_i^{m-1})).$[1]

2 On congruences on $n-$groups

2.1. Theorem *[Ušan 1997/1]: Let (Q, A) be an $n-$group,* **e** *its $\{1, n\}-$neutral operation, $^{-1}$ its inverse operation and $n \geq 3$. Let also θ be an equiv-*

[1] For $m = 2$, for example, in [Belousov 1967].

alence relation in the set Q such that for each $a, b, c_1^{n-1} \in Q$ the following formula holds

(0) $\bigwedge_{i=1}^{n} (a\ \theta\ b \Rightarrow A(c_1^{i-1}, a, c_i^{n-1})\ \theta\ A(c_1^{i-1}, b, c_i^{n-1}))$.[2]

Then, for each $a, b, c_1^{n-1} \in Q$ the following stetemnts hold:

(1) $\bigwedge_{i=1}^{n} (a\ \theta\ b \Leftrightarrow A(c_1^{i-1}, a, c_i^{n-1})\ \theta\ A(c_1^{i-1}, b, c_i^{n-1}))$;

(2) $a\ \theta\ b \Leftrightarrow (c_1^{n-2}, a)^{-1}\ \theta\ (c_1^{n-2}, b)^{-1}$;

(3) $\bigwedge_{i=1}^{n-2} (a\ \theta\ b \Leftrightarrow \mathbf{e}(c_1^{i-1}, a, c_i^{n-3})\ \theta\ \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))$; *and*

(4) $\bigwedge_{i=1}^{n-1} (a\ \theta\ b \Rightarrow (c_1^{i-1}, a, c_i^{n-2})^{-1}\ \theta\ (c_1^{i-1}, b, c_i^{n-2})^{-1})$.

Proof. 1) For each $a, b, c_1^{n-1} \in Q$ the following statement holds:

(0') $\bigwedge_{i=1}^{n} (A(c_1^{i-1}, a, c_i^{n-1})\ \theta\ A(c_1^{i-1}, b, c_i^{n-1}) \Rightarrow a\ \theta\ b)$.

The proof of (0'):

We shall consider, respectively, the cases: $i = 1$, $i = n$ and $i \in \{1, \ldots, n\} \setminus \{1, n\}$.

$i = 1$: By the assumption (0) and by Th. 1.3 from Chapter III, we have the following sequence of implications

$A(a, c_1^{n-1})\ \theta\ A(b, c_1^{n-1}) \Rightarrow$

$A(A(a, c_1^{n-1}), c_1^{n-2}, (c_1^{n-1})^{-1})\ \theta\ A(A(b, c_1^{n-1}), c_1^{n-2}, (c_1^{n-1})^{-1}) \Rightarrow a\ \theta\ b,$

hence

$A(a, c_1^{n-1})\ \theta\ A(b, c_1^{n-1}) \Rightarrow a\ \theta\ b.$

$i = n$: By the assumption (0) and by Th. 1.3 from Chapter III, we have the following implications:

$A(c_1^{n-1}, a)\ \theta\ A(c_1^{n-1}, b) \Rightarrow$

$A((c_2^{n-1}, c_1)^{-1}, c_2^{n-1}, A(c_1^{n-1}, a))\ \theta\ A((c_2^{n-1}, c_1)^{-1}, c_2^{n-1}, A(c_1^{n-1}, b)) \Rightarrow a\ \theta\ b,$

and thereby

$A(c_1^{n-1}, a)\ \theta\ A(c_1^{n-1}, b) \Rightarrow a\ \theta\ b.$

$i \in \{1, \ldots, n\} \setminus \{1, n\}$: By the assumption (0), and since (Q, A) is an $n-$semigroup, and also by (0') for $i = 1$ and $i = n$, we have the implications:

---

[2]See (b) in 1.

$$A(c_1^{i-1}, a, c_i^{n-1})\ \theta\ A(c_1^{i-1}, b, c_i^{n-1}) \Rightarrow$$
$$A(d_i^{n-1}, A(c_1^{i-1}, a, c_i^{n-1}), d_1^{i-1})\ \theta\ A(d_i^{n-1}, A(c_1^{i-1}, b, c_i^{n-1}), d_1^{i-1}) \Rightarrow$$
$$A(A(d_i^{n-1}, c_1^{i-1}, a), c_i^{n-1}, d_1^{i-1})\ \theta\ A(A(d_i^{n-1}, c_1^{i-1}, b), c_i^{n-1}, d_1^{i-1}) \Rightarrow$$
$$A(d_i^{n-1}, c_1^{i-1}, a)\ \theta\ A(d_i^{n-1}, c_1^{i-1}, b) \Rightarrow a\ \theta\ b,$$

and hence

$$A(c_1^{i-1}, a, c_i^{n-1})\ \theta\ A(c_1^{i-1}, b, c_i^{n-1}) \Rightarrow a\ \theta\ b.$$

Since the conjuction of (0) and (0') is equivalent with (1), we conclude that (1) holds.

2) By (just proved) statement (1), by Th. 1.3 from Chapter III, the following sequence of equivalences holds:

$$(c_1^{n-2}, a)^{-1}\ \theta\ (c_1^{n-2}, b)^{-1} \Leftrightarrow$$
$$A(a, c_1^{n-2}, (c_1^{n-2}, a)^{-1})\ \theta\ A(a, c_1^{n-2}, (c_1^{n-2}, b)^{-1}) \Leftrightarrow$$
$$A(A(a, c_1^{n-2}, (c_1^{n-2}, a)^{-1}), c_1^{n-2}, b)\ \theta\ A(A(a, c_1^{n-2}, (c_1^{n-2}, b)^{-1}), c_1^{n-2}, b) \Leftrightarrow$$
$$A(A(a, c_1^{n-2}, (c_1^{n-2}, a)^{-1}), c_1^{n-2}, b)\ \theta\ A(a, c_1^{n-2}, A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, b)) \Leftrightarrow$$
$$A(\mathbf{e}(c_1^{n-2}), c_1^{n-2}, b)\ \theta\ A(a, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \Leftrightarrow b\ \theta\ a$$

for all $a,\ b\ c_1^{n-2} \in Q$, and hence (2) holds.

3) By (1), by Proposition 1.3 from Chapter IV, and by (2), we have the following sequence of equivalences holds:

$$\begin{aligned}
a\ \theta\ b\ &\Leftrightarrow\ A(x, c_1^{i-1}, a, c_i^{n-3}, y)\ \theta\ A(x, c_1^{i-1}, b, c_i^{n-3}, y)\\
&\Leftrightarrow\ A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3}))^{-1}), a_1^{n-2}, y)\ \theta\\
&\qquad A(A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))^{-1}), a_1^{n-2}, y)\\
&\Leftrightarrow\ A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3}))^{-1})\ \theta\\
&\qquad A(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))^{-1})\\
&\Leftrightarrow\ (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3}))^{-1}\ \theta\ (a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))^{-1}\\
&\Leftrightarrow\ \mathbf{e}(c_1^{i-1}, a, c_i^{n-3})\ \theta\ \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}),
\end{aligned}$$

and hence, (3) holds.

4) By (0), by (3) and since

$$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}),$$

where E is a $\{1, 2n-1\}-$neutral operation of the $(2n-1)-$group $(Q, \overset{2}{A})$ /:Th. 1.3, Prop. 1.1 and Prop. 1.2 from Chapter III /, we have the following implications

$$a\ \theta\ b \ \Rightarrow\ \mathsf{E}(c_1^{i-1}, a, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3})\ \theta\ \mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3}) \text{ and}$$
$$a\ \theta\ b \ \Rightarrow\ \mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3})\ \theta\ \mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, b, c_i^{n-3})$$

for all all $a, b, c, c_1^{n-3} \in Q$, and hence we have the implication

$$a\ \theta\ b \Rightarrow \mathsf{E}(c_1^{i-1}, a, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3})\ \theta\ \mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, b, c_i^{n-3})$$ [3],

i.e.,

$$a\ \theta\ b \Rightarrow (c_1^{i-1}, a, c_i^{n-3}, c)^{-1}\ \theta\ (c_1^{i-1}, b, c_i^{n-3}, c)^{-1}$$

for every $i \in \{1, \dots, n-2\}$ and for every sequence a, b, c, c_1^{n-3} over a set Q, i.e. (4) holds. □

2.2 Remark: *Theorem 2.1 is proved under the assumption that $n \geq 3$. However, analizing the proof, one could easily see that (1) and (2) hold also for $n = 2$.*

3 On the set of all congruences of the given $n-$group, $n \geq 3$

3.1. Theorem *[Ušan 1998/2]: Let (Q, A) be an $n-$group, $n \geq 3$ and let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary nHG$-$algebra associated to the $n-$group (Q, A). Then, the following equality holds:*

$$Con(Q, A) = Con(Q, \cdot) \cap (Q, \varphi).$$

Proof. 1) $\Rightarrow$: Let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of an $n-$group (Q, A) [cf. Th. 2.6 from Chapter II], and let c_1^{n-2} be an arbitrary sequence over the set Q. Further on, let for every $x, y \in Q$ the following hold

(1) $\quad x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y)$,

(2) $\quad \varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$ and

(3) $\quad b \overset{def}{=} (\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|)$.

---

[3] $((p \Rightarrow q) \wedge (p \Rightarrow r)) \Leftrightarrow (p \Rightarrow q \wedge r)$.

Then $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to the $n-$group (Q, A) [cf. 2.3, 3.1 and 4.1 from Chapter IV]. In addition, for every $(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A$ [cf. Th. 4.1 from Chapter IV] there is a sequence c_1^{n-2} over Q such that for all $x, y \in Q$, (1)-(3). Further on, if $\theta \in Con(Q, A)$, since (1) and (2) hold for all $x, y \in Q$, we conclude that for all $x, y, \overline{x}, \overline{y} \in Q$ the following sequence of implications holds

$$\begin{array}{rcl} x\,\theta\,\overline{x} & \Rightarrow & A(x, c_1^{n-2}, y)\;\theta\; A(\overline{x}, c_1^{n-2}, y) \\ & & x \cdot y\;\theta\;\overline{x} \cdot y \\ y\,\theta\,\overline{y} & \Rightarrow & A(x, c_1^{n-2}, y)\;\theta\; A(x, c_1^{n-2}, \overline{y}) \\ & & x \cdot y\;\theta\; x \cdot \overline{y} \quad \text{and} \\ x\,\theta\,\overline{x} & \Rightarrow & A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})\;\theta\; A(\mathbf{e}(c_1^{n-2}), \overline{x}, c_1^{n-2}) \\ & \Rightarrow & \varphi(x)\;\theta\;\varphi(\overline{x}), \end{array}$$

whence we conclude that for an arbitrary $(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A$ and for arbitrary $\theta \in P(Q^2)$, the following implication holds

$$\theta \in Con(Q, A) \Rightarrow \theta \in Con(Q, \cdot) \wedge \theta \in Con(Q, \varphi).$$

2) $\Leftarrow$: Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q, A). Further on, let θ be an arbitrary element of the set $P(Q^2)$ such that the following conjunction holds

$$\theta \in Con(Q, \cdot) \wedge \theta \in Con(Q, \varphi).$$

Since $\theta \in Con(Q, \cdot)$, the following statement holds:

(a) for every $i \in \{1, \ldots, m\}$, $m \geq 2$, for every sequence a_1^m over the set Q and for all x, $\overline{x} \in Q$ the following implication holds

$$x\theta\overline{x} \Rightarrow (\prod_{j=1}^{i-1} a_j) \cdot x \cdot (\prod_{j=i}^{m-1} a_j)\;\theta\;(\prod_{j=1}^{i-1} a_j) \cdot \overline{x} \cdot (\prod_{j=i}^{m-1} a_j)^4.$$

Since $\theta \in Con(Q, \varphi)$, the following statement holds:

(b) for every $t \in N \cup 0$ and for all $x, \overline{x} \in Q$ the following implication holds

[4] $\prod_{j=p}^{p-1} a_j \overset{def}{=} e$, where e is the neutral element of the group $(Q, \cdot)$, and $p \in N$. See, also Chapter V-1.

$$x\theta\overline{x} \Rightarrow \varphi^t(x)\ \theta\ \varphi^t(\overline{x}).$$

Finally, by (a), (b) and by the assumption that $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to the $n-$group (Q, A), we conclude that for every $i \in \{1, \dots, n\}$, for every $x_1^n \in Q$ and for every $\overline{x}_1^n \in Q$ the following series of implications holds

$$\begin{aligned} x_i\theta\ \overline{x}_i &\Rightarrow \varphi^{i-1}(x_i)\ \theta\ \varphi^{i-1}(\overline{x}_i) \\ &\Rightarrow (\prod_{j=1}^{i-1} \varphi^{j-1}(x_j)) \cdot \varphi^{i-1}(x_i) \cdot (\prod_{j=i+1}^{n} \varphi^{j-1}(x_j)) \cdot b\ \theta \\ &\quad\ (\prod_{j=1}^{i-1} \varphi^{j-1}(x_j)) \cdot \varphi^{i-1}(\overline{x}_i) \cdot (\prod_{j=i+1}^{n} \varphi^{j-1}(x_j)) \cdot b \\ &\Rightarrow A(x_1^{i-1}, x_i, x_{i+1}^n)\ \theta\ A(x_1^{i-1}, \overline{x}_i, x_{i+1}^n). \quad \square \end{aligned}$$

3.2. Proposition: *Let* (Q, A) *be an* $n-$*group,* $n \geq 3$, *and let* $(Q, \{\cdot, \varphi, b\})$ *be an arbitrary* $nHG-$*algebra associated to the* $n-$*group* (Q, A)*. Also let* θ *be an arbitrary element of the set* $Con(Q, A)$*. Then for all* $x, y \in Q$ *the following equaivalence holds*

$$x\ \theta\ y \Leftrightarrow \varphi(x)\ \theta\ \varphi(y).$$

Proof. By Th. 2.1, by Th. 3.1 and by Th. 4.1 from Chapter IV, we conclude that for all $x, y \in Q$ the following sequence of equivalences holds

$$\begin{aligned} x\ \theta\ y &\overset{2.1}{\Leftrightarrow} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})\ \theta\ A(\mathbf{e}(c_1^{n-2}), y, c_1^{n-2}) \\ &\Leftrightarrow \varphi(x)\ \theta\ \varphi(y). \end{aligned}$$

$\square$

4 Construction of a lattice on a given $nHG-$ algebra

In the Theory of groups the following **three** propositions are well known.

4.1. Proposition: *Let* $(Q, \cdot)$ *be a group, and let*

$$L \overset{def}{=} \{H | (H, \cdot) \lhd (Q, \cdot)\}.$$

Also let

$H_1 \odot H_2 \stackrel{def}{=} \{x | x = h_1 \cdot h_2 \wedge h_1 \in H_1 \wedge h_2 \in H_2\}$ *and*
$H_1 \cap H_2 \stackrel{def}{=} \{x | x \in H_1 \wedge x \in H_2\}.$
Then $(L, \odot, \cap)$ *is a* **modular lattice.**

4.2. Proposition: *Let* $(Q, \cdot)$ *be a group and let* $^{-1}$ *be its inverse operation. Further on, let*
$L \stackrel{def}{=} \{H | (H, \cdot) \lhd (Q, \cdot)\}.$
Then, there is **exactly one bijection** F *of the set* $Con(Q, \cdot)$ *onto the set* L *such that for every* $\theta \in Con(Q, \cdot)$ *and for all* $x, y \in Q$ *the following statement holds*
$x \; \theta \; y \Leftrightarrow x \cdot y^{-1} \in F(\theta).$

4.3. Proposition: *Let* $(Q, \cdot)$ *be a group, let* $(L, \odot, \cap)$ *be a modular lattice from Prop. 4.1, and let* F *be a bijection of the set* $Con(Q, \cdot)$ *onto the set* L *from Prop. 4.2. Then,* F *is an* **isomorphism** *of the* $(Con(Q, \cdot), \circ, \cap)$ *to the lattice* $(L, \odot, \cap)$, *where*
$\theta_1 \circ \theta_2 \stackrel{def}{=} \{(x, y) | (\exists z \in Q)((x, z) \in \theta_1 \wedge (z, y) \in \theta_2)\}$ *and*
$\theta_1 \cap \theta_2 \stackrel{def}{=} \{(x, y) | (x, y) \in \theta_1 \wedge (x, y) \in \theta_2.\}$

4.4. Theorem: *[Ušan 1998/2]: Let* $(Q, \{\cdot, \varphi, b\})$ *be an* $nHG-$*algebra* $(n \geq 3)$ *and let* $^{-1}$ *be an inverse operation in the group* $(Q, \cdot)$. *Further on, let* $(H, \cdot) \lhd (Q, \cdot)$ *and let for every* $x, y \in Q$ *the following equivalence holds*
(0) $\quad x \; \theta \; y \Leftrightarrow x \cdot y^{-1} \in H \; [\theta \in Con(Q, \cdot)].$
Then the following statements are equivalent:
(1) $\quad \varphi(H)^5 = H,$
(2) $\quad \theta \in Con(Q, \varphi)$ *and*
(3) $\quad \varphi \in H!.$

Sketch of the proof.
1) (1) $\Rightarrow$ (2) :

---

[5] $\varphi(H) \stackrel{def}{=} \{\varphi(x) | x \in H\}.$

$$\begin{aligned} x\ \theta\ y &\overset{(0)}{\Leftrightarrow} x\cdot y^{-1}\in H \Leftrightarrow \varphi(x\cdot y^{-1})\in\varphi(H)\\ &\overset{(1)}{\Leftrightarrow} \varphi(x\cdot y^{-1})\in H \Leftrightarrow \varphi(x)\cdot\varphi(y^{-1})\in H\\ &\Leftrightarrow \varphi(x)\cdot(\varphi(y))^{-1}\in H \overset{(0)}{\Leftrightarrow} \varphi(x)\ \theta\ \varphi(y). \end{aligned}$$

2) $(2)\Rightarrow(3)$:

$$\begin{aligned} x\cdot y^{-1}\in H \overset{(0)}{\Leftrightarrow} x\ \theta\ y \overset{(2)}{\Leftrightarrow} \varphi(x)\ \theta\ \varphi(y) \\ \overset{(0)}{\Leftrightarrow}\varphi(x)\cdot(\varphi(y))^{-1}\in H, \end{aligned}$$ [6]

whence, we obtain

$$x\in H\Leftrightarrow\varphi(x)\in H.$$

Hence, by $\varphi\in Q!$, we conclude that the (3) holds.

3) $(3)\Rightarrow(1)$:

By

$$x\in H\Leftrightarrow\varphi(x)\in H$$

and by

$$x\in H\Leftrightarrow\varphi(x)\in\varphi(H),$$ [7]

we conclude that the following equivalence holds

$$\varphi(x)\in H\Leftrightarrow\varphi(x)\in\varphi(H)$$

for all $x\in Q$, i.e.

$$y\in H\Leftrightarrow y\in\varphi(H)$$

for all $y\in Q$ [$\varphi\in Q!$], whence we conclude that (1) holds. □

4.5. Theorem *[Ušan 1998/2]: Let $(Q,\{\cdot,\varphi,b\})$ be an $nHG-$algebra. Further on, let*

$$\widehat{L}\overset{def}{=}\{H|(H,\cdot)\lhd(Q,\cdot)\wedge\varphi(H)=H\}.$$

Also let

$$H_1\odot H_2\overset{def}{=}\{x|x=h_1\cdot h_2\wedge h_1\in H_1\wedge h_2\in H_2\}\ \textit{and}$$
$$H_1\cap H_2\overset{def}{=}\{x|x\in H_1\wedge x\in H_2\}.$$

Then $(\widehat{L},\odot,\cap)$ is a sublatittice of the modular lattice $(L,\odot,\cap)$ [from Prop. 4.1].

Proof. The following statements hold:

---

[6] See 3.2 and 3.1.

[7] See, the footnote [5].

1° For every $H_1, H_2 \in \widehat{L}$ the following equality holds

$$\varphi(H_1 \odot H_2) = H_1 \odot H_2; \text{ and}$$

2° For every $H_1, H_2 \in \widehat{L}$ the following equality holds

$$\varphi(H_1 \cap H_2) = H_1 \cap H_2.$$

Sketch of the proof of 1° :

$$\begin{aligned} \varphi(H_1 \odot H_2) &= \varphi\{h_1 \cdot h_2 | h_1 \in H_1 \wedge h_2 \in H_2\} \\ &= \{\varphi(h_1 \cdot h_2) | h_1 \in H_1 \wedge h_2 \in H_2\} \\ &= \{\varphi(h_1) \cdot \varphi(h_2) | h_1 \in H_1 \wedge h_2 \in H_2\} \\ &= \{\varphi(h_1) \cdot \varphi(h_2) | \varphi(h_1) \in \varphi(H_1) \wedge \varphi(h_2) \in \varphi(H_2)\} \\ &= \{\varphi(h_1) \cdot \varphi(h_2) | \varphi(h_1) \in H_1 \wedge \varphi(h_2) \in H_2\} \\ &= \{k_1 \cdot k_2 | k_1 \in H_1 \wedge k_2 \in H_2\} \\ &= H_1 \odot H_2. \end{aligned}$$

Sketch of the proof of 2° :

$$\begin{aligned} \varphi(x) \in \varphi(H_1 \cap H_2) &\Leftrightarrow x \in H_1 \cap H_2 \\ &\Leftrightarrow x \in H_1 \wedge x \in H_2 \\ &\Leftrightarrow \varphi(x) \in \varphi(H_1) \wedge \varphi(x) \in \varphi(H_2) \\ &\Leftrightarrow \varphi(x) \in H_1 \wedge \varphi(x) \in H_2 \\ &\Leftrightarrow \varphi(x) \in H_1 \cap H_2. \quad \square \end{aligned}$$

4.6. Example: *Let* $(\{1,2,3,4\},\cdot)$ *be the Klein group: Tab. 1. Further on, let* φ *be the permutation of the set* $\{1,2,3,4\}$ *defined in the following way*

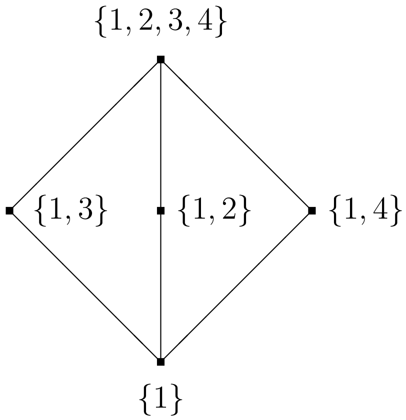
$$\varphi \overset{def}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

$\cdot$	*1*	*2*	*3*	*4*
1	*1*	*2*	*3*	*4*
2	*2*	*1*	*4*	*3*
3	*3*	*4*	*1*	*2*
4	*4*	*3*	*2*	*1*

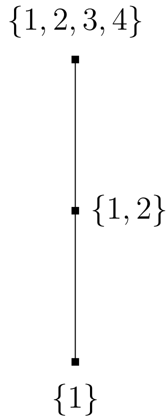
Tab. 1

Then $(\{1,2,3,4\},\{\cdot,\varphi,2\})$ *is a* $3HG-$*algebra*[8]. *In addition, the following holds:*
$L=\{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3,4\}\}$ *and* $\widehat{L}=\{\{1\},\{1,2\},\{1,2,3,4\}\}$;
$\varphi(\{1,3\})=\{1,4\}\neq\{1,3\},\ \varphi(\{1,4\})=\{1,3\}\neq\{1,4\}$.

Lattices $(L,\odot,\cap)$ *and* $(\widehat{L},\odot,\cap)$ *are represented in Diag. 1 and Diag. 2.*



Diag. 1



Diag. 2

4.7. Remark: *If* $(Q,\{\cdot,\varphi,b\})$ *is an* $nHG-$*algebra and* φ *an inner automorphism of the group* $(Q,\cdot)$, *then* $(\widehat{L},\odot,\cap)=(L,\odot,\cap)$. *However, there are* $nHG-$*algebras* $(Q,\{\cdot,\varphi,b\})$ *such that* φ *is not an inner automorphism of the group* $(Q,\cdot)$ *and* $(\widehat{L},\odot,\cap)=(L,\odot,\cap)$. *E.g.: Let* $(Q,\cdot)$ *be a commutative group,* $^{-1}$ *an inverse operation in* $(Q,\cdot)$ *and let there be at least one* $x\in Q$ *such that* $x^{-1}\neq x$. *Further on, let* $\varphi={}^{-1}$, $b=e$, *where* e *is the neutral element of the group* $(Q,\cdot)$. *Then* $(Q,\{\cdot,\varphi,b\})$ *is a* $3HG-$*algebra and* $\widehat{L}=L$.

By Th. 3.1 and by 4.1-4.5, we conclude that the following proposition holds:

[8]See, also Example 1.4 from Chapter I.

4.8. Proposition: *Let (Q, A) be an $n-$group, $n \geq 3$ and $(Q, \{\cdot, \varphi, b\})$ its arbitrary associated $nHG-$algebra. Further on, let*

$\widehat{F} = F$ *for each* $\theta \in Con(Q, \cdot) \cap Con(Q, \varphi)$,

where F from Prop. 4.2. Then, $\widehat{F}$ is an isomorphism of the $(Con(Q, A), \circ, \cap)$ to the lattice $(\widehat{L}, \odot, \cap)$.

4.9. Remark: *For every $n-$group (Q, A), $n \geq 3$, there is a group $(\overline{Q}, \cdot)$ and its normal subgroup $(H, \cdot)$ such that: 1) $Q \in \overline{Q}/H$; 2) the factor–group $(\overline{Q}/H, \square)$ [of the group $(Q, \cdot)$ over H] is a finite cyclic group; and 3) for every $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \ldots \cdot x_n$ [:Post's Coset theorem, 1940]. In [Monk, Sioson 1971] it was described the lattice of congruences of the $n-$group (Q, A), $n \geq 3$, up to an isomorphism, by means of the lattice of normal subgroups of the group $(H, \cdot)$ which are at the same time normal subgroup of the group $(\overline{Q}, \cdot)$. See, also [Janeva 1995].*

5 About congruence classes of $n-$groups

5.1. Example: *Let $(\{1, 2, 3, 4\}, \cdot)$ be Klein's group: Table 1. Then, $(\{1, 2, 3, 4\}, A)$, where*

$$A(x_1^3) \stackrel{def}{=} x_1 \cdot x_2 \cdot x_3 \cdot 3$$

for every $x_1^3 \in \{1, 2, 3, 4\}$, is a 3-group;

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 1.

Tables 2_1–2_4, $A_i(x, y) \stackrel{def}{=} A(x, i, y)$, $i \in \{1, 2, 3, 4\}$ [$A_1(x, y) = x \cdot y \cdot 3$, $A_2(x, y) = x \cdot y \cdot 4$, $A_3(x, y) = x \cdot y$, $A_4(x, y) = x \cdot y \cdot 2$].

A_1	1	2	3	4
1	3	4	1	2
2	4	3	2	1
3	1	2	3	4
4	2	1	4	3

Table 2_1

A_2	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Table 2_2

A_3	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 2_3

A_4	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

Table 2_4

The equivalence relation of the 3-group $(\{1,2,3,4\},A)$ *defined by the equality*

$$(\{1,2,3,4\}/\ \theta\ = \{\{1,2\},\{3,4\}\}$$

is a congruence relation of the 3-group $(\{1,2,3,4\},A)$ *(:Tables* $2_1 - 2_4$*).*

By Table $2_1 - 2_4$*, we conclude that the pairs* $(\{1,2\},A)$ *and* $(\{3,4\},A)$ **are not 3-groups**.

5.2. Example: *Let* $(\{1,2,3,4\},\cdot)$ *be Klein's group: Table 1. Then,* $(\{1,2,3,4\},B)$*, where* $B(x_1^3)\overset{def}{=}x_1\cdot x_2\cdot x_3\cdot 2$
for every $x_1^3\in\{1,2,3,4\}$*, is a 3-group; Table* 4_1-4_4*,* $B_i(x,y)\overset{def}{=}B(x,i,y)$ $i\in\{1,2,3,4\}$ *[:*$B_1(x,y)=x\cdot y\cdot 2,\ B_2(x,y)=x\cdot y,\ B_3(x,y)=x\cdot y\cdot 4,\ B_4(x,y)=x\cdot y\cdot 3$*].*

B_1	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

Table 4_1

B_2	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 4_2

B_3	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Table 4_3

B_4	1	2	3	4
1	3	4	1	2
2	4	3	2	1
3	1	2	3	4
4	2	1	4	3

Table 4_4

The equivalence relation θ *in the set* $\{1,2,3,4\}$ *defined by the equality*

$$\{1,2,3,4\}/\ \theta\ = \{\{1,2\},\{3,4\}\}$$

is a congruence relation of the 3-group $(\{1,2,3,4\},B)$ *(:Table* $4_1 - 4_4$*).*

By table $4_1 - 4_4$*, we conclude that the pairs* $(\{1,2\},B)$ *and* $(\{3,4\},B)$ **are** *3*-**groups**. *They are represented, respectively in Table* $6_1 - 6_2$ *and Table* $7_1 - 7_2$.

B_1	1	2
1	2	1
2	1	2

Table 6_1

B_2	1	2
1	1	2
2	2	1

Table 6_2

B_3	3	4
3	4	3
4	3	4

Table 7_1

B_4	3	4
3	3	4
4	4	3

Table 7_2

Examples 5.1 and 5.2 are from [Ušan 1998/1]. See, also [Janeva 1995].

5.3. Remark *[Janeva 1995]: There exists a 3-group (Q, A) and its congruence θ such that the following statements hold: a) $|Q/\theta| > 1$, and b)* **exactly one** *$\theta-$class is a 3-subgroup of the 3-group (Q, A).*

By 5.1–5.3, we conclude that the following proposition holds:

5.4. Proposition: *If $n \geq 3$, then: (a) there exist $n-$group (Q, A) and its congruence θ such that* **for every** *$C_a \in Q/\theta$ the pair (C_a, A)* **is not** *an $n-$group; (b) there exist an $n-$group (Q, A) and its congruence θ such that* **for every** *$C_a \in Q/\theta$ the pair (C_a, A)* **is** *an $n-$group; and (c) there exist $n-$group (Q, A) and its congruence θ such that* **exactly one** *$C_a \in Q/\theta$ the pair (C_a, A)* **is** *an $n-$group. [$|Q/\theta| \geq 2$.]*

5.5. Theorem *[Ušan 1998/2]: Let (Q, A) be an $n-$group and let $n \geq 3$. Further on, let θ be an arbitrary element of the set $Con(Q, A)$. Then,* **for every** *$C_t \in Q/\theta$* **there is** *an $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ associated to the $n-$group (Q, A) such that the following statements hold:*

(i) $(C_t, \cdot) \lhd (Q, \cdot)$,

(ii) (C_t, φ) is a 1-guasigroup, and

(iii) $(C_t, A)$is an $n-$subgroup of the $n-$group (Q, A) iff $b \in C_t$.

Proof. We prove that under the assumption the following statements hold:

1° For every $C_t \in Q/\theta$ there is a sequence c_1^{n-2} over Q such that

(0) $\quad \mathbf{e}(c_1^{n-2}) = t,$

where $\mathbf{e}$ is an $\{1, n\}-$neutral operation of the $n-$group (Q, A);

2° Let the sequence c_1^{n-2} over Q satisfies (0). Then the algebra $(Q, \{\cdot, \varphi, b\})$ defined by

(1) $\quad x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y),$

(2) $\quad \varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$ and

(3) $\quad b \overset{def}{=} A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|) \;\; [= A(\overset{n}{t})]$

is an $nHG-$algebra associated to the $n-$group (Q, A);

3° $(C_t, \cdot)$ is a subgroup of the group $(Q, \cdot)$;

4° $(C_t, \cdot) \lhd (Q, \cdot)$;

5° (C_t, φ) is a 1-quasigroup; and

6° The statement (iii) holds.

The proof of the statement 1° : By Th. 1.4 from Chapter IV.

The proof of the statement 2° : By Th. 4.1 from Chapter IV.

The proof of the statement 3° :

By Def. 2.1 from Chapter II and by 1°, we conclude that $\mathbf{e}(c_1^{n-2})$ is the neutral element of the group $(Q, \cdot)$, whence, by (0), we conclude that the neutral element $\mathbf{e}(c_1^{n-2})$ of the group $(Q, \cdot)$ belongs to C_t, i.e., that

(4) $\qquad \mathbf{e}(c_1^{n-2}) \in C_t.$

Further on, if $^{-1}$ is an inverse operation in the $n-$group (Q, A) [Th. 1.3 from Chapter III], then the unary operation $^{\mathbf{-1}}$ in Q, defined by

(5) $\qquad x^{\mathbf{-1}} \stackrel{def}{=} (c_1^{n-2}, x)^{-1},$

is an inverse operation in the group $(Q, \cdot)$. In addition, for every $\theta \in P(Q^2)$ the following implication holds

(6) $\qquad \theta \in Con(Q, A) \Rightarrow \theta \in Con(Q, ^{-1}) \qquad .$

[Th. 2.1 from Chapter VI.]

Finally, by 1° 2°, (4) – (6), 1 from VI, Th. 2.1 from VI and by Th. 1.3 from Chapter III, we conclude that for every $x, y \in Q$ the following series of implications holds

$$\begin{aligned}
x \in C_t \wedge y \in C_t &\Rightarrow x \,\theta\, \mathbf{e}(c_1^{n-2}) \wedge y \,\theta\, \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow (c_1^{n-2}, x)^{-1} \,\theta\, (c_1^{n-2}, \mathbf{e}(c_1^{n-2}))^{-1} \wedge y \,\theta\, \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow A((c_1^{n-2}, x)^{-1}, c_1^{n-2}, y) \,\theta\, A((c_1^{n-2}, \mathbf{e}(c_1^{n-2}))^{-1}, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \\
&\Rightarrow A((c_1^{n-2}, x)^{-1}, c_1^{n-2}, y) \,\theta\, \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow x^{\mathbf{-1}} \cdot y \,\theta\, \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow x^{\mathbf{-1}} \cdot y \in C_t,
\end{aligned}$$

whence, we conclude that $(C_t, \cdot)$ is a subgroup of the group $(Q, \cdot)$.

The proof of the statement 4° :

Let a be an arbitrary element from Q and let x be an arbitrary element from C_t. Then, by Th. 2.1, Th. 1.3 from Chapter III, 1°, 2° and (5), we conclude that the following series of equivalences holds

$$\begin{aligned} x \in C_t &\Leftrightarrow x \,\theta\, \mathbf{e}(c_1^{n-2}) \\ &\Leftrightarrow A(a, c_1^{n-2}, x) \,\theta\, A(a, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \\ &\Leftrightarrow A(a, c_1^{n-2}, x) \,\theta\, a \\ &\Leftrightarrow A(A(a, c_1^{n-2}, x), c_1^{n-2}, (c_1^{n-2}, a)^{-1}) \,\theta\, A(a, c_1^{n-2}, (c_1^{n-2}, a)^{-1}) \\ &\Leftrightarrow A(A(a, c_1^{n-2}, x), c_1^{n-2}, (c_1^{n-2}, a)^{-1}) \,\theta\, \mathbf{e}(c_1^{n-2}) \\ &\Leftrightarrow a \cdot x \cdot a^{-\mathbf{1}} \in C_t. \end{aligned}$$

The proof of the statement 5° : By (i), by Th. 3.1 and by Th. 4.4.

The proof of the statement 6° :

By 1° − 3° and by 5°, we conclude that for every $x_1^n \in C_t$ there is $y \in C_t$ such that the following equality holds

$$A(x_1^n) = y \cdot b.$$

Whence, by 3°, we conclude that (C_t, A) is an $n-$groupoid iff $b \in C_t$. Finally, hence, by 3° and by 5°, we conclude that the statement (iii) holds. □

6 On superpositions of an $n-$ary operations

6.1. Definition: *Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then:*

(i) $\overset{1}{A} \overset{def}{=} A$; and

(ii) for every $m \in N$ and for every $x_1^{(m+1)(n-1)+1} \in Q$

$$\overset{m+1}{A}(x_1^{(m+1)(n-1)+1}) \overset{def}{=} A(\overset{m}{A}(x_1^{m(n-1)+1}), x_{m(n-1)+2}^{(m+1)(n-1)+1}).$$

6.2. Proposition: *Let (Q, A) be an $n-$semigroup, $n \geq 2$ and $m \in N$. Then, for every $x_1^{(m+1)(n-1)+1} \in Q$ and for every $t \in \{1, \ldots, m(n-1)+1\}$ the following equality holds*

(1) $$\overset{m}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(m+1)(n-1)+1}) = \overset{m+1}{A}(x_1^{(m+1)(n-1)+1}).$$

Sketch of the proof.

1) $m = 1$: By Def. 6.1 and Def. 1.1 from Chapter I, we conclude that the following equality holds

$$A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}) = \overset{2}{A}(a_1^{2n-1})$$

for every $a_1^{2n-1} \in Q$ and for all $i \in \{1, \ldots, n\}$.

2) $m = v$: Let for every $a_1^{v(n-1)},\ b_1^n \in Q$ and for all $t \in \{1, \dots, v(n-1)+1\}$ the following equality holds

$$\overset{v}{A}(a_1^{t-1}, A(b_1^n), a_t^{v(n-1)}) = \overset{v+1}{A}(a_1^{t-1}, b_1^n, a_t^{v(n-1)}).$$

3) $v \to v+1$:

$$\overset{(v+1)+1}{A}(a_1^{(v+2)(n-1)+1}) \overset{(ii)}{=} A(\overset{v+1}{A}(a_1^{(v+1)(n-1)+1}), a_{(v+1)(n-1)+2}^{(v+2)(n-1)+1}) \overset{2)}{=}$$

$$A(\overset{v}{A}(a_1^{t-1}, A(a_t^{t+n-1}), a_{t+n}^{(v+1)(n-1)+1}), a_{(v+1)(n-1)+2}^{(v+2)(n-1)+1}) \overset{(ii)}{=}$$

$$\overset{v+1}{A}(a_1^{t-1}, A(a_t^{t+n-1}), a_{t+n}^{(v+2)(n-1)+1}) \overset{2)}{=}$$

$$\overset{v}{A}(a_1^{t-1}, A(A(a_t^{t+n-1}), a_{t+n}^{t+2(n-1)}), a_{t+2(n-1)+1}^{(v+2)(n-1)+1}) \overset{1.1-I}{=\!=}$$

$$\overset{v}{A}(a_1^{t-1}, A(a_t^{t+i-2}, A(a_{t+i-1}^{n+t+i-2}), a_{n+t+i-1}^{t+2(n-1)}), a_{t+2(n-1)+1}^{(v+2)(n-1)+1}) \overset{2)}{=}$$

$$\overset{v+1}{A}(a_1^{t-1}, a_t^{t+i-2}, A(a_{t+i-1}^{n+t+i-2}), a_{n+t+i-1}^{(v+2)(n-1)+1});$$

$t \in \{1, \dots, v(n-1)+1\},\ i \in \{1, \dots, n\}$. □

6.3. Theorem: *Let (Q, A) be an $n-$semigroup, $n \geq 2$ and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-1)+1} \in Q$ and for every $t \in \{1, \dots, i(n-1)+1\}$, the following equality holds*

(2) $$\overset{i}{A}(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-1)}), x_{t+j(n-1)+1}^{(i+j)(n-1)+1}) = \overset{i+j}{A}(x_1^{(i+j)(n-1)+1}).$$

Sketch of the proof.

1) For $j = 1$, $(2) = (1)$.

2) Let $j > 1$:

$$\overset{i}{A}(a_1^{t-1}, \overset{j}{A}(a_t^{t+j(n-1)}), a_{t+j(n-1)+1}^{(i+j)(n-1)+1}) \overset{(ii)}{=}$$

$$\overset{i}{A}(a_1^{t-1}, A(\overset{j-1}{A}(a_t^{t+(j-1)(n-1)}), a_{t+(j-1)(n-1)+1}^{t+j(n-1)}), a_{t+j(n-1)+1}^{(i+j)(n-1)+1}) \overset{6.2}{=}$$

$$\overset{i+1}{A}(a_1^{t-1}, \overset{j-1}{A}(a_t^{t+(j-1)(n-1)}), a_{t+(j-1)(n-1)+1}^{(i+j)(n-1)+1}).$$ □

6.4. Corollary: *Let (Q, A) be an $n-$semigroup [$n-$group], $n \geq 2$ and $i \in N$.*

Then, $(Q, \overset{i}{A})$ *is an* $(i(n-1)+1)-$*semigroup [*$(i(n-1)+1)-$*group].*[9] $\square$

See, also [Čupona 1969].

7 On the lattice of congruences on a class of polyadic groups

7.1. Theorem *[Ušan 1999/4]: Let* (Q, A) *be an* $n-$*group and let* $n \geq 2$*. Then for every* $k \in N$ *the following equality holds*

$$(Con(Q,A), \circ, \cap) = (Con(Q, \overset{k}{A}), \circ, \cap)^{10},$$

where $\overset{1}{A} \overset{def}{=} A$ *and* $\overset{m+1}{A}(x_1^{(m+1)(n-1)+1}) \overset{def}{=} A(\overset{m}{A}(x_1^{m(n-1)+1}), x_{m(n-1)+2}^{(m+1)(n-1)+1})^{11}$.

Proof. 1) $n \geq 3$: Firstly we prove the following statements:

1° If $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra [Chapter IV], then for every $k \in N$ $(Q, \{\cdot, \varphi, b^k\})$ is a $(k(n-1)+1)HG-$algebra; and

2° If $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to the $n-$group (Q, A), then for every $k \in N$ $(Q, \{\cdot, \varphi, b^k\})$ is a $(k(n-1)+1)HG-$algebra associated to the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$.

Sketch of the proof of 1° :

a) $\varphi(b^1) = b,\ \varphi(b^t) = b^t,$
$\varphi(b^{t+1}) = \varphi(b^t) \cdot \varphi(b) = b^t \cdot b = b^{t+1};$

b) $\varphi^{t(n-1)}(x) \cdot b^t = b^t \cdot x,$

$$\begin{aligned} \varphi^{(t+1)(n-1)}(x) \cdot b^{t+1} &= \varphi^{t(n-1)}(\varphi^{n-1}(x)) \cdot b^t \cdot b \\ &= b^t \cdot \varphi^{n-1}(x) \cdot b \\ &= b^t \cdot b \cdot x \\ &= b^{t+1} \cdot x. \end{aligned}$$

Sketch of the proof of 2° :

---

[9] See Th. 6.3 for $i = j$ and Def. 1.1 from Chapter I.

[10] See Prop. 4.8.

[11] See 6. from VI.

$\overline{a})$ $\overset{1}{A} = A,\ A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b;$

$\overline{b})$ $\overset{t}{A}(x_1^{t(n-1)+1}) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{t(n-1)}(x_{t(n-1)+1}) \cdot b^t;$

$\overline{c})$ $\overset{t+1}{A}(x_1^{(t+1)(n-1)+1}) = A(x_1^{n-1}, \overset{t}{A}(x_n^{(t+1)(n-1)+1})) =$
$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot \varphi^{n-1}(\overset{t}{A}(x_n^{(t+1)(n-1)+1})) \cdot b =$
$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot \varphi^{n-1}(x_n) \cdot \ldots \cdot \varphi^{(t+1)(n-1)}(x_{(t+1)(n-1)+1}) \cdot b^t \cdot b =$
$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{(t+1)(n-1)}(x_{(t+1)(n-1)+1}) \cdot b^{t+1}.$

By 1°, 2° and Th. 3.1 from VI, we conclude that for every $k \in N$ the following equality holds

(1) $\quad Con(Q, \overset{k}{A}) = Con(Q, A).$

2) $n = 2$: Let (Q, A) be a group. If $A = \cdot$, then for every $x_1^{k+1} \in Q$ the following equality holds

$$\overset{k}{A}(x_1^{k+1}) = x_1 \cdot \ldots \cdot x_{k+1}.$$

Whence, for $k \geq 2$, we conclude that $(Q, \{\cdot, I, \mathsf{e}\})$ is an $nHG-$algebra associated to the $(k+1)-$group $(Q, \overset{k}{A})$; $I \overset{def}{=} \{(x, x) | x \in Q\}$, e is a neutral element in $(Q, A)[= (Q, \cdot)]$. Finally, by Th. 3.1 from VI $[k + 1 \geq 3]$, we conclude that for every $k \in N$ the following equality holds

$$Con(Q, \overset{k}{A}) = Con(Q, A)[= Con(Q, \cdot)]. \quad \square$$

Chapter VII

ON ORDERED $n-$GROUPS

1 Ordered $n-$groups and ordered $nHG-$algebras

1.1. Definition *[Crombez 1972]: Let (Q, A) be an $n-$group, $n \geq 2$ and let $\leq$ be a partial order on Q. Let also for all $x, y, z_1^{n-1} \in Q$ and for all $i \in \{1, 2, \ldots, n\}$ the following implication holds*

(1) $x \leq y \Rightarrow A(z_1^{i-1}, x, z_i^{n-1}) \leq A(z_1^{i-1}, y, z_i^{n-1})$.

Then, we say that $(Q, A, \leq)$ is an **ordered** $n-$**group**.

Note that in the case $n = 2$ $(Q, A, \leq)$ is an ordered group in the sense of [Fusch 1963].

1.2. Theorem *[Ušan, Žižović 1997]: Let $\leq$ be a partial order on Q. Also, let (Q, A) be an $n-$group and let $n \geq 3$. In addition, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q, A). Then, $(Q, A, \leq)$ is an ordered $n-$group iff for all $x, y, z \in Q$ the following two, formulas hold*

(2) $x \leq y \Rightarrow x \cdot z \leq y \cdot z \wedge z \cdot x \leq z \cdot y$

(3) $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$.

Proof. 1) Let $(Q, A, \leq)$ be an ordered $n-$group and let $n \geq 3$. Also, let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of the $n-$group (Q, A). In addition, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q, A). Then, by Th. 4.1 from Chapter IV, there is at least one sequence c_1^{n-2} over Q such that for every $x, y \in Q$ the following two equalities hold:

$$x \cdot y = A(x, c_1^{n-2}, y),$$
$$\varphi(x) = A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}).$$

Hence, by Def. 1.1, we conclude that the formulas (2) and (3) hold in $(Q, \{\cdot, \varphi, b\})$.

2) Conversely, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q, A). Also, let $\leq$ be a partial order in Q. Assume that an $nHG-$algebra $(Q, \{\cdot, \varphi, b\})$ satisfies (2) and (3). Then, for every $x, y, z_1^{n-2} \in Q$

2_1) for $i \in \{2, \ldots, n-1\}$ the following series of implications holds

$x \leq y \overset{(3)}{\Rightarrow} \varphi^{i-1}(x) \leq \varphi^{i-1}(y) \overset{(2)}{\Rightarrow}$
$z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(x) \leq z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(y) \overset{(2)}{\Rightarrow}$
$z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(x) \cdot \varphi^{i}(z_i) \cdot \ldots \cdot b \cdot z_{n-1} \leq$
$z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(y) \cdot \varphi^{i}(z_i) \cdot \ldots \cdot b \cdot z_{n-1},$

hence, by Definition 2.3 from Chapter IV, we conclude that the following implication holds

$$x \leq y \Rightarrow A(z_1^{i-1}, x, z_i^{n-1}) \leq A(z_1^{i-1}, y, z_i^{n-1});$$

2_2) for $i = 1$ the following implication holds

$$x \leq y \overset{(2)}{\Rightarrow} x \cdot \varphi(z_1) \cdot \ldots \cdot b \cdot z_{n-1} \leq y \cdot \varphi(z_1) \cdot \ldots \cdot b \cdot z_{n-1},$$

hence, by Definition 2.3 from Chapter IV, we conclude that the following implication holds

$$x \leq y \Rightarrow A(x, z_1^{n-1}) \leq A(y, z_1^{n-1}); \text{ and}$$

2_3) for $i = n$ the following implication hold

$$x \leq y \overset{(2)}{\Rightarrow} z_1 \cdot \varphi(z_2) \cdot \ldots \cdot \varphi^{n-2}(z_{n-1}) \cdot b \cdot x \leq z_1 \cdot \varphi(z_2) \cdot \ldots \cdot \varphi^{n-2}(z_{n-1}) \cdot b \cdot y,$$

whence, by Definition 2.3 from Chapter IV, we conclude that the following implication holds

$$x \leq y \Rightarrow A(z_1^{n-1}, x) \leq A(z_1^{n-1}, y). \quad \square$$

1.3. Example: *Let $(Z, +)$ be the additive group of all integers, and $\leq$ the natural order defined on Z. Then Z with the ternary operation A defined by*

$$A(x, y, z) = x + (-y) + z$$

is a 3-group.

Moreover, $(Z, \{+, \varphi, 0\})$, where $\varphi(x) = -x$, is an $nHG-$algebra associated to the $3-$group (Z, A).

Since for every $x, y \in Z$ $x \leq y$ implies $\varphi(y) \leq \varphi(x)$, we conclude, by Theorem 1.2, that $(Z, A, \leq)$ is not an ordered $3-$group.

1.4. Example: *Let $(Z, +, \leq)$ be as in the previous example. Let*

$$B(x_1^n) \overset{def}{=} x_1 + x_2 + \ldots + x_n + 2$$

for every $x_1^n \in Z$, $n \geq 3$. Then, (Z, B) is an $n-$group with $(Z, \{+, I, 2\})$, where $I \overset{def}{=} \{(x, x) | x \in Z\}$, as its associated nHG-algebra. Obviously $(Z, B, \leq)$ is an ordered $n-$group.

Moreover, $(Z, C, \leq)$ and $(Z, D, \leq)$ where

$$C(x_1^n) \overset{def}{=} x_1 + x_2 + \ldots + x_n,$$

$$D(x_1^n) \overset{def}{=} x_1 + x_2 + \ldots + x_n + (-2)$$

are ordered $n-$groups as well. $\square$

2 Two propositions more

2.1. Theorem *[Ušan, Žižović 1997]: Let $(Q, A, \leq)$ be an ordered $n-$group and let $n \geq 2$. Also, let $^{-1}$ be an inverse operation of the $n-$group (Q, A). Then, for every $x, y, z_1^{n-1} \in Q$ the following stetements hold*

(1) $\bigwedge_{i=1}^{n} (x \leq y \Leftrightarrow A(z_1^{i-1}, x, z_i^{n-1}) \leq A(z_1^{i-1}, y, z_i^{n-1}))$, *and*

(2) $x \leq y \Leftrightarrow (z_1^{n-2}, y)^{-1} \leq (z_1^{n-2}, x)^{-1}$.

Proof.

The proof of the stetement (1):

a) $\Rightarrow$: By Def. 1.1.

b) $\Leftarrow$: In the case $i = 1$, by Def. 1.1, we conclude

$$A(x, a_1^{n-2}, a) \leq A(y, a_1^{n-2}, a) \Rightarrow$$
$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \leq A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Rightarrow$$
$$A(x, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) \leq A(y, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) \Rightarrow$$
$$A(x, a_1^{n-2}\mathbf{e}(a_1^{n-2})) \leq A(y, a_1^{n-2}\mathbf{e}(a_1^{n-2})) \Rightarrow x \leq y.$$

The case $i = n$ may be proved analogously.

Let now $i \in \{2, \ldots, n-1\}$. Then

$A(a_1^{i-1}, x, a_i^{n-1}) \leq A(a_1^{i-1}, y, a_i^{n-1}) \Rightarrow$
$A(b_i^{n-1}, A(a_1^{i-1}, x, a_i^{n-1}), b_1^{i-1}) \leq A(b_i^{n-1}, A(a_1^{i-1}, y, a_i^{n-1}), b_1^{i-1}) \Rightarrow$
$A(A(b_i^{n-1}, a_1^{i-1}, x), a_i^{n-1}, b_1^{i-1}) \leq A(A(b_i^{n-1}, a_1^{i-1}, y), a_i^{n-1}, b_1^{i-1}) \overset{"i=1"}{\Longrightarrow}$
$A(b_i^{n-1}, a_1^{i-1}, x) \leq A(b_i^{n-1}, a_1^{i-1}, y) \overset{"i=n"}{\Longrightarrow} x \leq y.$

Sketch of the proof of the statement (2):

$$\begin{aligned} x \leq y \ &\overset{(1)}{\Leftrightarrow}\ A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, x) \leq A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, y) \\ &\Leftrightarrow\ \mathbf{e}(a_1^{n-2}) \leq A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, y) \\ &\Leftrightarrow\ A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, y)^{-1}) \leq \\ &\qquad A(A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, y), a_1^{n-2}, (a_1^{n-2}, y)^{-1}) \\ &\Leftrightarrow\ (a_1^{n-2}, y)^{-1} \leq A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, A(y, a_1^{n-2}, (a_1^{n-2}, y)^{-1})) \\ &\Leftrightarrow\ (a_1^{n-2}, y)^{-1} \leq A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) \\ &\Leftrightarrow\ (a_1^{n-2}, y)^{-1} \leq (a_1^{n-2}, x)^{-1}. \quad \square \end{aligned}$$

2.2. Theorem *[Ušan, Žižović 1997]: Let $(Q, A, \leq)$ be an ordered $n-$group and let $n \geq 3$. Also, let $^{-1}$ be an inverse operation of the $n-$group (Q, A), and let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of (Q, A). Then, for every $x, y, b, a_1^{n-3} \in Q$ the following stetements hold*

(3) $\bigwedge_{i=1}^{n-2} (x \leq y \Leftrightarrow \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}) \leq \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}));$ *and*

(4) $\bigwedge_{i=1}^{n-2} (x \leq y \Rightarrow (a_1^{i-1}, y, a_i^{n-3}, b)^{-1} \leq (a_1^{i-1}, x, a_i^{n-3}, b)^{-1}).$

Proof.

The proof of the statement (3):

Since

(5) $A(a, b_1^{n-2}, b) = A(A(a, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}(b_1^{n-2}))^{-1}), a_1^{n-2}, b)$

by Prop. 1.3 from Chapter IV, then

$$\begin{aligned} x \leq y \ &\overset{(1)}{\Longleftrightarrow}\ A(a, a_1^{i-1}, x, a_i^{n-3}, b) \leq A(a, a_1^{i-1}, y, a_i^{n-3}, b) \\ &\overset{(5)}{\Longleftrightarrow}\ A(A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1}), c_1^{n-2}, b) \leq \\ &\qquad A(A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1}), c_1^{n-2}, b) \end{aligned}$$

$$\overset{(1)}{\Longleftrightarrow} \quad A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1}) \leq A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1})$$

$$\overset{(1)}{\Longleftrightarrow} \quad (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1} \leq (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1}$$

$$\overset{(2)}{\Longleftrightarrow} \quad \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}) \leq \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}).$$

The proof of the statement (4):

Let E be an $\{1, 2n-1\}-$neutral operation of the $(2n-1)-$group $(Q, \overset{2}{A})$; cf. Chapter III and 6 from Chapter VI. Hence, by 1.1 and by (3), we conclude

$$x \leq y \Rightarrow \mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, y, a_i^{n-3}) \leq \mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3})$$

and

$$x \leq y \Rightarrow \mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}) \leq \mathsf{E}(a_1^{i-1}, x, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}).$$

Whence, by transitivity of $\leq$, we conclude

$$x \leq y \Rightarrow \mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, y, a_i^{n-3}) \leq \mathsf{E}(a_1^{i-1}, x, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}).$$

This completes the proof because

$$(a_1^{i-1}, z, a_i^{n-3}, b)^{-1} \overset{def}{=} \mathsf{E}(a_1^{i-1}, z, a_i^{n-3}, b, a_1^{i-1}, z, a_i^{n-3});$$

cf. Chapter III. $\square$

3 On left and right cone

3.1. Theorem *[Ušan, Žižović 1977]: Let $(Q, A, \leq)$ be an ordered $n-$group, $n \geq 3$ and let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of the $n-$group (Q, A). Also, let $^{-1}$ be an inverse operation of (Q, A). Moreover, let a be an arbitrary element of the set Q and let a_1^{n-2} be an sequence over Q such that $\mathbf{e}(a_1^{n-2}) = a$. Then:*

- (i) *$(\{x | a \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A) iff $a \leq A(\overset{n}{a})$;*

- (ii) *If $(\{x | (a_1^{n-2}, A(\overset{n}{a}))^{-1} \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A) and $(Q, A, \leq)$ is a linearly ordered $n-$group, then $A(\overset{n}{a}) \leq a$;*

- $(\widehat{ii})$ *If $A(\overset{n}{a}) \leq a$, then $(\{x | (a_1^{n-2}, A(\overset{n}{a}))^{-1} \leq x\}, A)$ is an $n-$subsemigroup* *of the $n-$group (Q, A);*

(iii) Let $a \leq A(\overset{n}{a})$ and let c be an arbitrary element of the set Q such that $a \leq c$. Then $(\{x|c \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A); and

(iv) Let $A(\overset{n}{a}) \leq a$ and let c be an arbitrary element of the set Q such that $(a_1^{n-2}, A(\overset{n}{a}))^{-1} \leq c$. Then $(\{x|c \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A).

Proof. 1) Let a be an arbitrary element of the set Q. Also let a_1^{n-2} be an sequence over Q such that $\mathbf{e}(a_1^{n-2}) = a$ [cf. Prop. 1.4 from Chapter IV]. Moreover, let

(a) $x \cdot y \overset{def}{=} A(x, a_1^{n-2}, y)$,

(b) $\varphi(x) \overset{def}{=} A(a, x, a_1^{n-2})$,

(c) $b \overset{def}{=} A(\overset{n}{a})$ and

(d) $x^{\mathbf{-1}} \overset{def}{=} (a_1^{n-2}, x)^{-1}$

for all $x, y \in Q$. Then:

1° $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to (Q, A) [cf. Th. 4.1 from Chapter IV];

2° $a = \mathbf{e}(a_1^{n-2})$ is a neutral element of the group $(Q, \cdot)$; and

3° $^{\mathbf{-1}}$ is an inverse operation of the group $(Q, \cdot)$.

By Th. 1.2 and 1°, we conclude that

4° $(Q, \cdot, \leq)$ is an ordered group; and

5° $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$ for all $x, y \in Q$.

2) The proof of (i) :

a) $\Rightarrow$: Assume now that $(\{x|a \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A). Then for all $x_1^n \in Q$ from $x_1^n \in \{x|a \leq x\}$ it follows $A(x_1^n) \in \{x|a \leq x\}$, whence we conclude that $a \leq A(\overset{n}{a})$.

b) $\Leftarrow$: Let $a \le A(\overset{n}{a})$. Hence, by $1^\circ,\ 4^\circ$ and 5°, we conclude that for every sequence x_1^n over $\{x|a \le x\}$ the following two formulas hold:

$$a \le a \cdot \varphi(a) \cdot \ldots \cdot \varphi^{n-1}(a) \cdot b$$

and

$$a \cdot \varphi(a) \cdot \ldots \cdot \varphi^{n-1}(a) \cdot b \le x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \quad ^1.$$

Whence, by transitivity of $\le$, we conclude

$$a \le x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$$

for all $x_1^n \in \{x|a \le x\}$, i.e. by 1°, for all $x_1^n \in \{x|a \le x\}$,

$$A(x_1^n) \in \{x|a \le x\}.$$

So $(\{x|a \le x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A).

3) The proof of (ii) :

Assume now that $(\{x|b^{-\mathbf{1}} \le x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A); $b = A(\overset{n}{a})$, (d), 3°. Then for all $x_1^n \in Q$ from $x_1^n \in \{x|b^{-\mathbf{1}} \le x\}$ it follows $b^{-\mathbf{1}} \le A(x_1^n)$, whence, by $b^{-\mathbf{1}} \le b^{-\mathbf{1}}$, 1° and $\varphi(b^{-\mathbf{1}}) = b^{-\mathbf{1}}$, we conclude that

$$\begin{aligned} b^{-\mathbf{1}} &\le A(b^{-\mathbf{1}}, b^{-\mathbf{1}}, \ldots, b^{-\mathbf{1}}) \\ &= b^{-\mathbf{1}} \cdot \varphi(b^{-\mathbf{1}}) \cdot \ldots \cdot \varphi^{n-2}(b^{-\mathbf{1}}) \cdot b \cdot b^{-\mathbf{1}} \\ &= b^{-\mathbf{1}}, b^{-\mathbf{1}}, \ldots, b^{-\mathbf{1}}, \end{aligned}$$

i.e. $b^{n-2} \le a$. Hence $b \le a$ by $(Q, \cdot, \le)$ is a linearly ordered group [4° and $(Q, A, \le)$ is a linearly ordered $n-$group[2]].

4) The proof of $(\widehat{ii})$:

Let $b \le a$; $b = A(\overset{n}{a}) - (c)$. Then, by 4°, we have $a \le b^{-\mathbf{1}}$. Whence, by 1°, 2°, 4°, 5° and $\varphi(b^{-\mathbf{1}}) = b^{-\mathbf{1}}$, for all $x_1^n \in \{x|b^{-\mathbf{1}} \le x\}$, we obtain

$$\begin{aligned} b^{-\mathbf{1}} &\le b^{-\mathbf{1}} \cdot b^{-\mathbf{1}} = b^{-\mathbf{1}} \cdot \varphi(b^{-\mathbf{1}}) \cdot \varphi^2(a) \cdot \ldots \cdot \varphi^{n-2}(a) \cdot b \cdot b^{-\mathbf{1}} \\ &\le b^{-\mathbf{1}} \cdot \varphi(b^{-\mathbf{1}}) \cdot \varphi^2(b^{-\mathbf{1}}) \cdot \ldots \cdot \varphi^{n-2}(b^{-\mathbf{1}}) \cdot b \cdot b^{-\mathbf{1}} \\ &\le x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n \\ &= A(x_1^n), \end{aligned}$$

i.e. $b^{-\mathbf{1}} \le A(x_1^n)$ for all $x_1^n \in \{x|b^{-\mathbf{1}} \le x\}$. So $(\{x|b^{-\mathbf{1}} \le x\}, A)$ is an $n-$subsemigroup of the $n-$group (Q, A).

---

[1] $x \le y \wedge u \le v \Rightarrow x \cdot u \le y \cdot v.$

[2] $\neg(x \le y) \Leftrightarrow y < x,\ (p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p),\ b^{n-2} \le a.$

5) Sketch of the proof of (iii) :

$$\begin{array}{rcl} c &=& c\cdot\varphi(a)\cdot\ldots\cdot\varphi^{n-1}(a)\cdot a \leq c\cdot\varphi(c)\cdot\ldots\cdot\varphi^{n-1}(c)\cdot a \\ &\leq& c\cdot\varphi(c)\cdot\ldots\cdot\varphi^{n-1}(c)\cdot b \\ &\leq& x_1\cdot\varphi(x_2)\cdot\ldots\cdot\varphi^{n-1}(x_n)\cdot b \\ &=& A(x_1^n); \end{array}$$

$a \leq b,\ b = A(\overset{n}{a}),\ a \leq c,\ a = \mathbf{e}(a_1^{n-2})\ -\ 2^\circ$, footnote [1], $x_1^n \in \{x|c \leq x\}$.

6) Sketch of the proof of (iv) :

$$\begin{array}{rcl} c &=& c\cdot a\cdot\ldots\cdot a\cdot b\cdot b^{-\mathbf{1}} = c\cdot\varphi(a)\cdot\ldots\cdot\varphi^{n-2}(a)\cdot b\cdot b^{-\mathbf{1}} \\ &\leq& c\cdot\varphi(b^{-\mathbf{1}})\cdot\ldots\cdot\varphi^{n-2}(b^{-\mathbf{1}})\cdot b\cdot b^{-\mathbf{1}} \\ &\leq& c\cdot\varphi(c)\cdot\ldots\cdot\varphi^{n-2}(c)\cdot b\cdot c \\ &\leq& x_1\cdot\varphi(x_2)\cdot\ldots\cdot\varphi^{n-2}(x_{n-1})\cdot b\cdot x_n \\ &=& A(x_1^n); \end{array}$$

$b \leq a \Leftrightarrow a \leq b^{-\mathbf{1}}, b^{-\mathbf{1}} \leq c,\ x_1^n \in \{x|c \leq x\}$. □

3.2. Remark: *The above theorem describes so–called the* **right cone** *(cf. [Fuchs 1963]), i.e. the set* $K_R(c) \overset{def}{=} \{x|c \leq x\}$. *The analogous result holds for the* **left cone** $K_L(c) \overset{def}{=} \{x|x \leq c\}$. □

Chapter VIII

ON TOPOLOGICAL $n-$GROUPS

1 Introduction

1.1. Definition: *Let (Q,F) be an $m-$groupoid, $m \geq 1$ and let Q be equipped with a topology $\mathcal{O}$. Then, the $m-$ary operation F is continuous in $\mathcal{O}$ iff for every $x_1^m \in Q$ the following statement holds*

$$(\forall O_{F(x_1^m)} \in \mathcal{O})(\exists O_{x_i} \in \mathcal{O})_1^m F(O_{x_1}, \ldots, O_{x_m}) \subseteq O_{F(x_1^m)}.^1$$

1.2. Remark: *Let (Q,F) be an $m-$groupoid, $m \in N$ and let S_i be a subset of Q and $S_i \neq \emptyset$. Moreover, let*

$$\mathsf{F}(S_1^m) \stackrel{def}{=} \bigcup_{(x_1^m) \in S_1 \times \ldots \times S_m} \{F(x_1^m)\}.$$

Howeover, instead of F, usually, we write F.

1.3. Definition *[Ušan 1998/4]: Let (Q,A) be an $n-$group, $^{-1}$ its inverse operation (Chapter III), $n \geq 2$ and let Q be equipped with a topology $\mathcal{O}$. Then, we say that $(Q,A,\mathcal{O})$ is a topological $n-$group iff*

(1) The $n-$ary operation A is continuous in $\mathcal{O}$; and

(2) The $(n-1)-$ary operation $^{-1}$ is continuous in $\mathcal{O}$.[2]

1.4. Remark: *Topological $n-$groups have been defined in mutually different ways. Each of these definitions was different from the description given in Definition 1.3. In addition, the definitions in [Čupona 1971], [Enders 1995] and [Ušan 1998/4] are related to each $n \geq 2$, while those in [Crombez, Six 1974] and [Žižović 1976] are restricted (only) to each $n \geq 3$. All definitions*

---

[1] $O_z \in \mathcal{O}$ and $z \in O_z$.

[2] For $n = 2$ $(Q,A,\mathcal{O})$ is a topological group [:e.g. [Pontryagin 1973]].

from the cited papers are mutually equivalent for $n \geq 3$, *while the definitions from the papers [Čupona 1971], [Enders 1995] and [Ušan 1998/4] are also mutually equivalent for* $n = 2$ [*:[Rusakov 1992], [Ušan 1998/4]*].

2 Auxiliary propositions

2.1. Proposition *[Ušan 1997/4, 1999/6]: Let* (Q, A) *be an* $n-$*group,* $^{-1}$ *its inverse operation,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* $n \geq 2$. *Also let*

(*l*) $^{-1}A(x, a_1^{n-2}, y) = z \overset{def}{\Longleftrightarrow} A(z, a_1^{n-2}, y) = x$ *and*

(*r*) $A^{-1}(x, a_1^{n-2}, y) = z \overset{def}{\Longleftrightarrow} A(x, a_1^{n-2}, z) = y$

for all $x, y, z \in Q$ *and for every sequence* a_1^{n-2} *over* Q. *Then, for all* $x, y \in Q$ *and for every sequence* a_1^{n-2} *over* Q *the following equalities hold*

(1_l) $^{-1}A(x, a_1^{n-2}, y) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1})$,

(2_l) $\mathbf{e}(a_1^{n-2}) = {}^{-1}A(x, a_1^{n-2}, x)$,

(3_l) $(a_1^{n-2}, x)^{-1} = {}^{-1}A({}^{-1}A(x, a_1^{n-2}, x), a_1^{n-2}, x)$,

(4_l) $A(x, a_1^{n-2}, y) = {}^{-1}A(x, a_1^{n-2}, {}^{-1}A({}^{-1}A(y, a_1^{n-2}, y), a_1^{n-2}, y))$,

(1_r) $A^{-1}(x, a_1^{n-2}, y) = A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, y)$,

(2_r) $\mathbf{e}(a_1^{n-2}) = A^{-1}(x, a_1^{n-2}, x)$,

(3_r) $(a_1^{n-2}, x)^{-1} = A^{-1}(x, a_1^{n-2}, A^{-1}(x, a_1^{n-2}, x))$ *and*

(4_r) $A(x, a_1^{n-2}, y) = A^{-1}(A^{-1}(x, a_1^{n-2}, A^{-1}(x, a_1^{n-2}, x)), a_1^{n-2}, y)$.

Proof. The proof of (1_l) :

By (l), by Def. 1.1 from Chapter I and by Th. 1.3 from Chapter III, we conclude that the following series of equivalences holds

$$^{-1}A(x, a_1^{n-2}, y) = z \overset{(l)}{\Longleftrightarrow} A(z, a_1^{n-2}, y) = x \Leftrightarrow$$
$$A(A(z, a_1^{n-2}, y), a_1^{n-2}, (a_1^{n-2}, y)^{-1}) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1}) \Leftrightarrow$$
$$A(z, a_1^{n-2}, A(y, a_1^{n-2}, (a_1^{n-2}, y)^{-1})) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1}) \Leftrightarrow$$
$$A(z, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1}) \Leftrightarrow$$
$$z = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1})$$

for all $x, y, z \in Q$ and for every sequence a_1^{n-2} over Q, and hence

$${}^{-1}A(x,a_1^{n-2},y)=A(x,a_1^{n-2},(a_1^{n-2},y)^{-1}).$$

The proof of (2_l) :
By (l), we conclude that the following equivalence holds
${}^{-1}A(x,a_1^{n-2},x)=\mathbf{e}(a_1^{n-2})\Leftrightarrow A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$
for all $x\in Q$ and for every sequence a_1^{n-2} over Q. Whence, by Th. 2.6 from Chapter II, we conclude that (2_l) holds.

Sketch of the proof of (3_l) :
${}^{-1}A({}^{-1}A(x,a_1^{n-2},x),a_1^{n-2},x)=(a_1^{n-2},x)^{-1}\overset{(l)}{\Longleftrightarrow}$
$A((a_1^{n-2},x)^{-1},a_1^{n-2},x)={}^{-1}A(x,a_1^{n-2},x)\overset{(2_l)}{\Longleftrightarrow}$
$A((a_1^{n-2},x)^{-1},a_1^{n-2},x)=\mathbf{e}(a_1^{n-2}).$

Sketch of the proof of (4_l) :
$A(x,a_1^{n-2},y)={}^{-1}A(x,a_1^{n-2},{}^{-1}A({}^{-1}A(y,a_1^{n-2},y),a_1^{n-2},y)\overset{(l),(3_l)}{\Longleftrightarrow}$
$x=A(A(x,a_1^{n-2},y),a_1^{n-2},(a_1^{n-2},y)^{-1})\Longleftrightarrow$
$x=A(x,a_1^{n-2},A(y,a_1^{n-2},(a_1^{n-2},y)^{-1}))\Longleftrightarrow$
$x=A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})).$

Similarly, it is possible to prove also the equalities $(1_r)-(4_r)$ for all $x,y\in Q$ and for every sequence a_1^{n-2} over Q. $\square$

By Def. 1.1, we conclude that the following two propositions hold.

2.2. Proposition: *Let (Q,f) be an $n-$groupoid, let (Q,g) be an $m-$groupoid, let $m,n\in N$ and let Q be equipped with a topology $\mathcal{O}$. Also, let f and g be continuous in $\mathcal{O}$, and let*

$$F(x_1^m,y_1^{n-1})\overset{def}{=}f(y_1^{i-1},g(x_1^m),y_i^{n-1}),\ i\in\{1,\dots,n\},$$

for all $x_1^m,y_1^{n-1}\in Q$. Then, F is continuous in $\mathcal{O}$.

2.3. Proposition: *Let (Q,f) be an $n-$groupoid, $n\geq 2$ and let Q be equipped with a topology $\mathcal{O}$. Also, let F be continuous in $\mathcal{O}$, and let*

$$F(x_1^{n-1})\overset{def}{=}f(x_1^{i-1},c,x_i^{n-1}),\ i\in\{1,\dots,n\},$$

for all $x_1^{n-1}\in Q$, where c is a (fixed) element of the set Q. Then, F is continuous in $\mathcal{O}$.

$[f(O_{x_1},\dots,O_{x_{i-1}},\{c\},O_{x_i},\dots,O_{x_{n-1}})\subseteq O_{f(x_1^{i-1},c,x_i^{n-1})}.]$

2.4. Proposition: *Let (Q,F) be an $m-$groupoid, $m\geq 2$ and let Q be equipped with a topology $\mathcal{O}$. Also, let the $m-$ary operation F be continuous in $\mathcal{O}$, and let*

(0) $\Phi(x,a_1^{m-2})\overset{def}{=}F(x,a_1^{m-2},x)$

for all $x\in Q$ and for every sequence a_1^{m-2} over Q. Then $(m-1)-$ary operation Φ is continuous in $\mathcal{O}$.

Proof. Let

$$(\forall O_{F(x_1^m)}\in\mathcal{O})(\exists O_i\in\mathcal{O})_1^m\ F(O_{x_1},\dots,O_{x_m})\subseteq O_{F(x_1^m)}.$$

Whence

(1) $(\forall O_{F(x,a_1^{m-2},x)}\in\mathcal{O})(\exists\overline{O}_x\in\mathcal{O})(\exists\overline{\overline{O}}_x\in\mathcal{O})(\exists O_{a_i}\in\mathcal{O})_1^{m-2}$
$F(\overline{O}_x,O_{a_1},\dots,O_{a_{m-2}},\overline{\overline{O}}_x)\subseteq O_{F(x,a_1^{m-2},x)}.$

Also let

(2) $O_x\overset{def}{=}\overline{O}_x\cap\overline{\overline{O}}_x.$

By (0), (1) and (2), we conclude that the following formula holds

$$(\forall O_{\Phi(x,a_1^{m-2})}\in\mathcal{O})(\exists O_x\in\mathcal{O})(\exists O_{a_i}\in\mathcal{O})_1^{m-2}\Phi(O_x,O_{a_1},\dots,O_{a_{m-2}})\subseteq O_{\Phi(x,a_1^{m-2})}.$$

Similarly, it is posible to prove also, for example, the **three following propositions**. □

2.5. Proposition: *Let (Q,f) be an $(m+r+s+t)$-groupoid, let $m\geq 1$, $r+s+t\geq 2$ and $r,s,t\in N\cup\{0\}$. Also, let Q be equipped with a topology $\mathcal{O}$, and let f be continuous in $\mathcal{O}$. Further on, let*

$$F(x,y_1^m)\overset{def}{=}f(\overset{r}{x},y_1^{i-1},\overset{s}{x},y_i^m,\overset{t}{x})$$

for all $x,y_1^m\in Q$, $i\in\{1,\dots,m+1\}$. Then, F is continuous in $\mathcal{O}$.

2.6. Proposition: *Let (Q,f) be a $(r+t+1)$-groupoid, (Q,g) be a $(r+t+1)$-groupoid, (Q,h) be a $(s+t)$-groupoid, let $t\in N$ and let $r,s\in N\cup\{0\}$. Also, let Q be equipped with a topology $\mathcal{O}$, and let f,g and h be continuous in $\mathcal{O}$. Further on, let*

$$F(x_1^s,a_1^t,y_1^r)\overset{def}{=}f(h(x_1^s,a_1^t),a_1^t,y_1^r),$$

$\Phi(x_1^r, a_1^t, y_1^s) \overset{def}{=} g(x_1^r, a_1^t, h(y_1^s, a_1^t)),$
$\widehat{F}(x_1^s, a, y_1^r) \overset{def}{=} f(h(x_1^s, \overset{t}{a}), \overset{t}{a}, y_1^r)$ *and*
$\widehat{\Phi}(x_1^r, a, y_1^s) \overset{def}{=} g(x_1^r, \overset{t}{a}, h(y_1^s, \overset{t}{a})).$
Then $F, \Phi, \widehat{F}$ *and* $\widehat{\Phi}$ *are continuous in* $\mathcal{O}$.

2.7. Proposition: *Let* (Q, f) *be a* $(3+s)-$*groupoid,* $s \in N$, *let* g *be a* $t-$*groupoid and* $t \geq 1$. *Also, let* Q *be equipped with a topology* $\mathcal{O}$, *and let* f *and* g *be continuous in* $\mathcal{O}$. *Further on, let*

$$F(x, a_1^t) \overset{def}{=} f(g(a_1^t), \overset{i-1}{x}, g(\overset{t}{x}), \overset{s-i+1}{x}, g(a_1^t))$$

for all $x, a_1^t \in Q$; $i \in \{1, \ldots, s+1\}$. *Then,* F *is continuous in* $\mathcal{O}$.

2.8. Proposition *[Ušan 1998/4]: Let* (Q, A) *be an* $n-$*group,* $^{-1}$ *its inverse operation (Chapter III),* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation, and* $n \geq 3$. *Then for all* $a, a_1^{n-2} \in Q$ *the following equality holds*
$(a_1^{n-2}, a)^{-1} = A(\mathbf{e}(a_1^{n-2}), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})).$

Proof. By Def. 1.1 from Chapter I, and by Th. 3.1 from Chapter III, we conclude that for all $a_1^{n-2}, a \in Q$ the following sequence of equivalences holds
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \Leftrightarrow$
$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})) =$
$A(\mathbf{e}(a_1^{n-2}), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})) \Leftrightarrow$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-3}, A(a_{n-2}, a, \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a})), \mathbf{e}(a_1^{n-2})) =$
$A(\mathbf{e}(a_1^{n-2}), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})) \Leftrightarrow$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-3}, a_{n-2}, \mathbf{e}(a_1^{n-2})) =$
$A(\mathbf{e}(a_1^{n-2}), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})) \Leftrightarrow$
$(a_1^{n-2}, a)^{-1} = A(\mathbf{e}(a_1^{n-2}), \overset{n-3}{a}, \mathbf{e}(\overset{n-2}{a}), \mathbf{e}(a_1^{n-2})).$
Whence, we conclude that the proposition is satisfied. □

2.9. Proposition *[Žižović 1998]: Let* (Q, A) *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation,* $n \geq 4$*, and let*

(1) $\quad \overline{a} \stackrel{def}{=} \mathbf{e}(\overset{n-2}{a})$[3]

for all $a \in Q$*. Then for every sequence* a_1^{n-2} *over* Q *the following equality holds*

(2) $\quad \mathbf{e}(a_1^{n-2}) = \overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a_{n-2}}, \ldots, \overline{a}_1, \overset{n-3}{a_1}).$

Proof. By (1), by Def. 6.1 and by Prop. 6.3 from Chapter VI, and by Th. 2.6 from Chapter II, we conclude that for every sequence a_1^{n-2} over Q and for all $x \in Q$ the following series of equalities hold

$$A(\overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a_{n-2}}, \ldots, \overline{a}_1, \overset{n-3}{a_1}), a_1^{n-2}, x) =$$
$$A(\overset{n-3}{A}(\mathbf{e}(\overset{n-2}{a_{n-2}}), \overset{n-3}{a_{n-2}}, \ldots, \mathbf{e}(\overset{n-2}{a_1}), \overset{n-3}{a_1}), a_1^{n-2}, x) =$$

\- -

\- -

$$A(\mathbf{e}(\overset{n-2}{a_{n-2}}), \overset{n-3}{a_{n-2}}, a_{n-2}, x) = x,$$

whence, by Def. 2.1 from Chapter II, by Th. 2.6 from Chapter II and by (1), we conclude that for every sequence a_1^{n-2} over Q and for all $x \in Q$ the following equality holds

$$A(\overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a_{n-2}}, \ldots, \overline{a}_1, \overset{n-3}{a_1}), a_1^{n-2}, x) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x).$$

Whence, by Def. 1.1 from Chapter I, we conclude that for every sequence a_1^{n-2} over Q the equality (2) holds. □

3 Main propositions

3.1. Theorem *[Ušan 1999/1]: Let* (Q, A) *be an* $n-$*group,* $^{-1}$ *its inverse operation (Chapter III),* $n \geq 2$ *and let* Q *be equipped with a topology* $\mathcal{O}$*. Also let*

(0) $\quad {}^{-1}A(x, a_1^{n-2}, y) = z \stackrel{def}{\Leftrightarrow} A(z, a_1^{n-2}, y) = x$ *and*

[3]Actually, the unary operation **skew element** is in question; Apendix 2. For $n = 3$ $\overline{a} = \mathbf{e}(a)$.

$(\widehat{0})$ $\qquad A^{-1}(x, a_1^{n-2}, y) = z \overset{def}{\Leftrightarrow} A(x, a_1^{n-2}, z) = y$

for all $x, y, z \in Q$ *and for every sequence* a_1^{n-2} *over* Q*. Then the following stetements are equivalent:*

(i) *the* $n-$*ary operation* A *is continuous in* $\mathcal{O}$ *and the* $(n-1)-$*ary operation* $^{-1}$ *is continuous in* $\mathcal{O}$ *[cf. Def. 1.3];*

(ii) *the* $n-$*ary operation* ^{-1}A *is continuous in* $\mathcal{O}$; *and*

(iii) *the* $n-$*ary operation* A^{-1} *is continuous in* $\mathcal{O}$.

Proof. 1) Let Q be equipped with topology $\mathcal{O}$. Also, let the $n-$ary operation A be continuous in $\mathcal{O}$ and the $(n-1)-$ary operation $^{-1}$ be continuous in $\mathcal{O}$. Then, by $(1_l)[(1_r)]$ from 2.1, and by Prop. 2.6, we conclude that the $n-$ary operation $^{-1}A[A^{-1}]$ is continuous in $\mathcal{O}$.

2) Let Q be equipped with topology $\mathcal{O}$. Also, let the $n-$ary operation $^{-1}A[A^{-1}]$ be continuous in $\mathcal{O}$. Then by (3_l) and (4_l) $[(3_r)$ and $(4_r)]$ from Prop. 2.1, and by Prop. 2.4, and by Prop. 2.6, we conclude that the $n-$ary operation A is continuous in $\mathcal{O}$ and the $(n-1)-$ary operation $^{-1}$ is continuous in $\mathcal{O}$. □

3.2. Remark: *In [Čupona 1971],* $(Q, A, \mathcal{O})$ *is a topological* $n-$*group iff following statements hold:* (a) *the* $n-$*ary operation* A *is continuous* $\mathcal{O}$, (b) *the* $n-$*ary operation* ^{-1}A *is continuous in* $\mathcal{O}$, *and* (c) *the* $n-$*ary operation* A^{-1} *is continuous in* $\mathcal{O}$*. S. A. Ruskov has proved (1984) [cf. [Rusakov 1992]] that the* $n-$*group* (Q, A) *is topological with respect to the topology* $\mathcal{O}$ *iff* A *and* ^{-1}A *or* A *and* A^{-1} *are continuous in the topology* $\mathcal{O}$.

3.3. Theorem *[Ušan 1998/4]: Let* (Q, A) *be an* $n-$*group,* $n \geq 3$, *and let* $(Q, \{\cdot, \varphi, b\})$ *be an arbitrary* $nHG-$*algebra associated to the* $n-$*group* (Q, A) *[cf. Chapter III].Also, let* Q *be aquipped with a topology* $\mathcal{O}$*. Then,* $(Q, A, \mathcal{O})$ *is a topological* $n-$*group iff the following statements hold*

(1) $(Q, \cdot, \mathcal{O})$ *is a topological group; and*

(2) the unary operation φ *is continuous in* $\mathcal{O}$.

Proof. 1) $\Rightarrow$: Let $(Q, A, \mathcal{O})$ be a topological $n-$group (Def. 1.3), and let $n \geq 3$. Also, let $\mathbf{e}$ and $^{-1}$, respectively, be an $\{1, n\}-$neutral operation and an inverse operation of the $n-$group (Q, A). In addition, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group (Q, A). Then, by Th. 4.1 from Chapter IV, there is at least one sequence c_1^{n-2} over Q such that for every $x, y \in Q$ the following equalities hold
(3) $x \cdot y = A(x, c_1^{n-2}, y)$; and
(4) $\varphi(x) = A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$.
Let $^{\mathbf{-1}}$ be an inverse operation of the group $(Q, \cdot)$. By (3), Th. 3.1 from Chapter III, and by Th. 4.1 from Chapter IV, we conclude that for every $x \in Q$ the following equality holds
(5) $x^{\mathbf{-1}} = (c_1^{n-2}, x)^{-1}$;
$\mathbf{e}(c_1^{n-2})$ is a neutral element of the group $(Q, \cdot)$.

Finally, by (3)-(5), and by Prop. 2.3, we conclude that the statements (1) and (2) hold.

2) $\Leftarrow$: Let $(Q, \{\cdot, \varphi, b\})$ be an $nHG-$algebra associated to the $n-$group (Q, A). So, for all $x_1^n \in Q$ the following equality holds
(6) $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n$.
Whence, by Th. 1.3 from Chapter III, and by Prop. 4.2 from Chapter IV, we conclude that for all $x_1^{n-2}, x \in Q$ the following equality holds
(7) $(x_1^{n-2}, x)^{-1} = (\varphi(x_1) \cdot \ldots \cdot \varphi^{n-2}(x_{n-2}) \cdot b)^{\mathbf{-1}} \cdot x^{\mathbf{-1}} \cdot (\varphi(x_1) \cdot \ldots \cdot \varphi^{n-2}(x_{n-2}) \cdot b)^{\mathbf{-1}}$.
Also let Q be equipped with a topology $\mathcal{O}$. Further on, let the statements (1) and (2) hold.

Finally, by (1),(2),(6) and (7), and by Prop. 2.2, we conclude that the $(Q, A, \mathcal{O})$ is a topological $n-$group. □

3.4. Remark: *In [Crombez 1971] and [Žižović 1976],* $(Q, A, \mathcal{O})$ *is a topological $n-$group, $n \geq 3$, iff the following statements hold: (a) the $n-$ary operation A is continuous in $\mathcal{O}$, and (b) the unary operation on $^{-}$ (operation "skew element", cf. Appendix 2) is continuous in $\mathcal{O}$.*

4 Three more propositions

4.1. Proposition *[Ušan 1998/4]: Let* (Q, A) *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation,* $^{-1}$ *its inverse operation, and* $n \geq 3$*. Also, let* Q *be equipped with a topology* $\mathcal{O}$*. Further on, let the* $n-$*ary operation* A *be continuous in* $\mathcal{O}$*. Then the following statements are equivalent:*

(i) *the* $(n-1)-$*ary operation* $^{-1}$ *is continuous in* $\mathcal{O}$*; and*

(ii) *the* $(n-2)-$*ary operation* $\mathbf{e}$ *is continuous in* $\mathcal{O}$*.*

Proof. a) $(i) \Rightarrow (ii)$: By Th. 1.3 from Chapter III, we conclude that for all $a_1^{n-2} \in Q$ the following equality holds

$$\mathbf{e}(a_1^{n-2}) = A((a_1^{n-2}, a_1)^{-1}, a_1^{n-2}, a_1).$$

Whence, by Prop. 2.5, and by Prop. 2.6, we conclude the statement $(i) \Rightarrow (ii)$ is satisfied.

b) $(ii) \Rightarrow (i)$: By Prop. 2.8 and by Prop. 2.7. □

Similarly, it is posible to prove also the following proposition:

4.2. Proposition *[Ušan 1998/4]: let* (Q, A) *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation,* $^{-1}$ *its inverse operation, and* $n \geq 3$*. Also, let* $\overline{a} \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$ *and* $a^{(-2)} \overset{def}{=} (\overset{n-1}{a})^{-1}$*. Further on, let* Q *be equipped with a topology* $\mathcal{O}$*, and let the* $n-$*ary operation* A *be continuous in* $\mathcal{O}$*. Then the following statements are equivalent:*

$(\bar{i})$ *the unary operation* $^{-}$ *is continuous in* $\mathcal{O}$*; and*

$(\overline{ii})$ *the unary operation* $^{(-2)}$ *is continuous in* $\mathcal{O}$*.*

4.3. Remark: *In [Enders 1995],*$(Q, A, \mathcal{O})$ *is a topological* $n-$*group,* $n \geq 2$*, iff the following statements hold: 1) the* $n-$*ary operation* A *is continuous in* $\mathcal{O}$*, and 2) the unary operation* $^{(-2)}$ *is continuous in* $\mathcal{O}$*. For* $n = 2$*,* $a^{(-2)} = a^{-1}$.

4.4. Proposition *[Ušan 1998/4]: Let* (Q, A) *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation,* $n \geq 3$*, and let*

(1) $\overline{a} \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$[4]

for all $a \in Q$. Also, let Q be equipped a topology $\mathcal{O}$. Further on, let the $n-$ary operation A be continuous in $\mathcal{O}$. Then the following statements are equivalent:

$(\widehat{i})$ *the $(n-2)-$ary operation $\mathbf{e}$ is continuous in $\mathcal{O}$; and*

$(\widehat{ii})$ *the unary operation $^{-}$ is continuous in $\mathcal{O}$.*

Proof. 1) $(\widehat{i}) \Rightarrow \widehat{ii})$: By (1) and By Prop. 2.5.
2) $(\widehat{ii}) \Rightarrow \widehat{i})$: By Prop. 2.9 and by Def. 1.1. (See, also the proof of Prop. 2.4.) □

---

[4]See the footnote up the Proposition 2.9.

Chapter IX

SOME MORE CHARACTERIZATIONS OF $n-$GROUPS

1 $n-$groups as algebras of the type $< n, n-1 >$ with laws

1.1. Theorem *[Ušan 1994]: Let $n \geq 2$ and $(Q; A)$ be an $n-$semigroup. Then, (Q, A) is an $n-$group iff there is a mapping $^{-1}$ of the set Q^{n-1} into the set Q such that the laws (4_L) and (4_R) from Chapter III-3 holds in the algebra $(Q, \{A,^{-1}\})$.*[1]

This result is improved in [Dudek 1995]:

1.2. Theorem *[Dudek 1995]: Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then: (Q, A) is an $n-$group iff there is a mapping $^{-1}$ of the set Q^{n-1} into the set Q such that the following laws hold in the algebra $(Q, \{A,^{-1}\})$ [of the type $< n, n-1 >$]*

(1) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$[2],

(2) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$ *and*

(3) $A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x$.[3]

Proof.[4] 1)⇒: By Def. 1.1 from Chapter I and Theorem 1.3 from Chapter III.

---

[1] See, also Prop. 1.1 from Chapter III.
[2] or: $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$.
[3] In Chapter III: $(1) = (1_L), (2) = (4_L), (3) = (4_R)$.
[4] [Ušan 1997/2].

2) $\Leftarrow$: Firstly we prove the following statements:

1° For every $a, a_1^{n-2}, x, y \in Q$ the implication

$$A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow x = y$$

holds;

2° (Q, A) is an $n-$semigroup;

3° For every $a, a_1^{n-2}, x, y \in Q$ the implication

$$A(a, a_1^{n-2}, x) = A(a, a_1^{n-2}, y) \Rightarrow x = y$$

holds; and

4° For every $a_1^n \in Q$ there is exactly one x and exactly one $y \in Q$ such that the following equalities hold

$$A(a_1^{n-1}, x) = a_n \text{ and } A(y, a_1^{n-1}) = a_n.$$

Sketch of the proof of 1° :

$A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow$
$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(3)}{\Rightarrow} x = y.$

The proof of the statement 2° :
By 1° and by Prop. 2.1 from Chapter III.

The proof of the statement 3° : By (2).

Sketch of the proof of 4° :

a) $A(x, a_1^{n-2}, a) = b \overset{1^\circ}{\Leftrightarrow}$
$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(3)}{\Leftrightarrow}$
$x = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}).$

b) $A(a, a_1^{n-2}, x) = b \overset{3^\circ}{\Leftrightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = A((a_1^{n-2} a)^{-1}, a_1^{n-2}, b) \overset{(2)}{\Leftrightarrow}$
$x = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b).$

Finally, by $2^\circ, 4^\circ$ and by Prop.2.2 from Chapter III, we conclude that (Q, A) is an $n-$group.

Remark: Similarly, it is possible to prove the case "$(1_R), (4_L), (4_R)$". See footnotes [2] and [3]. □

1.3. Theorem *[Ušan 1997/2]: Let* (Q, A) *be an* $n-$*groupoid and* $n \geq 2$. *Then there is* **at most one** *mapping* $^{-1}$ *of the set* Q^{n-1} *into the set* Q *such that the laws (1)-(3) from Theorem 1.2 hold in the algebra* $(Q, \{A,^{-1}\})$ *of the type* $< n, n-1 >$.

Proof. Assume that there are mappings

$$^{-1_1}\ :\ Q^{n-1} \to Q \text{ and } ^{-1_2}\ :\ Q^{n-1} \to Q$$

such that the laws (1)-(3) from Theorem 1.2 hold in the algebras $(Q, \{A,^{-1_1}\})$ and $(Q, \{A,^{-1_2}\})$. Whence, by Th. 1.2, we conclude that the following statement holds:

1^* (Q, A) is an $n-$group.

Further on, by (2), we conclude that the following statement holds

2^* for all $a, x \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

$$A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x \text{ and}$$
$$A((a_1^{n-2}, a)^{-1_2}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x.$$

Finally, by 1^*, 2^* and by Def. 1.1 from Chapter I, we conclude that for all $a \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$(a_1^{n-2}, a)^{-1_1} = (a_1^{n-2}, a)^{-1_2},$$

i.e., that $^{-1_1} =^{-1_2}$. □

Remark: Similarly, it is possible to prove the case "$(1_R), (4_L), (4_R)$". See footnotes [2] and [3].

1.4. Theorem *[Ušan 1997/2]: The laws (1)-(3) from Theorem 1.2 are mutually independent.*

Proof. a) The laws (1) and (2) from Th. 1.2 hold in the algebra $(Q, \{A,^{-1}\})$ of the type $< n, n-1 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \stackrel{def}{=} x_n$, and $(a_1^{n-2}, a)^{-1} \stackrel{def}{=}$ $c-$constant. However, the law (3) from Th. 1.2 does not hold.

b) The laws (1) and (3) from Th. 1.2 hold in the algebra $(Q, \{A,^{-1}\})$ of the type $< n, n-1 >$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_1$, and $(a_1^{n-2}, a)^{-1} \overset{def}{=}$ $c-$constant. However, the law (2) from Th. 1.2 does not hold.

c_1) The case $n > 2$: Let $(Q, \square)$ be a group, $^{\mathbf{-1}}$ its inverse operation, and let (Q, B) be an $(n-2)-$groupoid which is not an $(n-2)-$quasigroup. Then (Q, A), where

$$A(x, a_1^{n-2}, y) \overset{def}{=} x \square (B(a_1^{n-2}))^{\mathbf{-1}} \square y,$$

satisfies conditions of Proposition 1.2 from Chapter III. Thus, there is an algebra $(Q, \{A, ^{-1}\})$ of the type $< n, n-1 >$, in which the laws (2) and (3) from Th. 1.2 hold. However, the law (1) fails to hold in $(Q, \{A, ^{-1}\})$. Indeed, if the law (1) from Th. 2.1 holds in $(Q, \{A, ^{-1}\})$, then by Th. 2.1 is an $n-$group, which contradict the assumption that (Q, B) is not an $(n-2)-$guasigroup.

c_2) The case $n = 2$: Let (Q, A) be a Moufang loop which is not a group, and let $^{-1}$ its inverse operation. Then the laws (2) and (3) from Th. 1.2 hold in the algebra $(Q, \{A, ^{-1}\})$. However, the law (1) does not hold. (Cf. [Bruck 1958] or [Belousov 1967].) $\square$

Remark: Similarly, it is possible to prove the case "$(1_R), (4_L), (4_R)$". See footnotes [2)] and [3)].

2 $n-$groups, $n \geq 3$, as algebras of the type $< n, n-2 >$ with laws

2.1. Theorem *[Ušan 1988]: Let $n \geq 3$ and let (Q, A) be an $n-$semigroup. Then: (Q, A) is an $n-$group iff (Q, A) has an $\{1, n\}-$neutral operation.*[5]

This result was improved in [Ušan 1997/2]:

2.2. Theorem *[Ušan 1997/2]: Let (Q, A) be an $n-$groupoid and $n \geq 3$. Then: (Q, A) is an $n-$group iff there is a mapping* $\mathbf{e}$ *of the set Q^{n-2} into the*

[5]See, also Th. 2.6 from Chapter II.

set Q such that the following laws hold in the algebra $(Q,\{A,\mathbf{e}\})$ [of the type $<n,n-2>$]

$(\bar{1})$ $A(A(x_1^n),x_{n+1}^{2n-1})=A(x_1,A(x_2^{n-1}),x_{n+2}^{2n-1})$[6],

$(\bar{2})$ $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$ *and*

$(\bar{3})$ $A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))=x$.[7]

Proof. 1)$\Rightarrow$: By Def. 1.1 from Chapter I and Th. 1.3 from Chapter III.

2) $\Leftarrow$: Firstly we prove the following statements:

$^\circ 1$ For every $a,a_1^{n-2},x,y\in Q$ the implication holds

$A(x,a,a_1^{n-2})=A(y,a,a_1^{n-2})\Rightarrow x=y$;

$^\circ 2$ (Q,A) is an $n-$semigroup;

$^\circ 3$ For every $a,a_1^{n-2},x,y\in Q$ the implication holds

$A(a_1^{n-2},a,x)=A(a_1^{n-2},a,y)\Rightarrow x=y$; and

$^\circ 4$ For every $a_1^n\in Q$ there is exactly one x and exactly one $y\in Q$ such that the following equalities hold

$$A(a_1^{n-1},x)=a_n \text{ and } A(y,a_1^{n-1})=a_n.$$

Sketch of the proof of $^\circ 1$:

$A(x,a,a_1^{n-2})=A(y,a,a_1^{n-2})\Rightarrow$

$A(A(x,a,a_1^{n-2}),\mathbf{e}(a_1^{n-2}),c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))=$

$A(A(y,a,a_1^{n-2}),\mathbf{e}(a_1^{n-2}),c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))\overset{(\bar{1})}{\Rightarrow}$

$A(x,A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})),c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))=$

$A(y,A(a,a_1^{n-2},\mathbf{e}(a_1^{n-2})),c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))\overset{(\bar{3})}{\Rightarrow}$

$A(x,a,c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))=A(y,a,c_1^{n-3},\mathbf{e}(a,c_1^{n-3})\overset{(\bar{3})}{\Rightarrow}$

$x=y$.

The proof of the statement $^\circ 2$:

By $^\circ 1$ and by Prop. 2.1 from Chapter III.

Sketch of the proof of $^\circ 3$:

$A(a_1^{n-2},a,x)=A(a_1^{n-2},a,y)\Rightarrow$

---

[6]or: $A(x_1^{n-2},A(x_{n-1}^{2n-2}),x_{2n-1})=A(x_1^{n-1},A(x_n^{2n-1}))$.

[7]In Chapter III-3: $(\bar{1})=(1_L)$, $(\bar{2})=(2_L)$, $(\bar{3})=(2_R)$.

$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(a_1^{n-2}),A(a_1^{n-2},a,x)) =$
$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(a_1^{n-2}),A(a_1^{n-2},a,y))\overset{\circ 2}{\Rightarrow}$
$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),x) =$
$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},a),y)\overset{(\overline{2})}{\Rightarrow}$
$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,x) = A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,y)\overset{(\overline{2})}{\Rightarrow}$
$x = y.$

Sketch of the proof of $\circ 4$:

a) $A(x,a,a_1^{n-2}) = b\overset{\circ 1}{\Longleftrightarrow}$
$A(A(x,a,a_1^{n-2}),\mathbf{e}(a_1^{n-2}),c_1^{n-3},\mathbf{e}(a,c_1^{n-3})) = A(b,\mathbf{e}(a_1^{n-2}),c_1^{n-3},\mathbf{e}(a,c_1^{n-3}))$
$\overset{(\overline{1}),(\overline{3})}{\Longleftrightarrow} x = A(b,\mathbf{e}(a_1^{n-2}),c_1^{n-3},\mathbf{e}(a,c_1^{n-3})).$

b) $A(a_1^{n-2},a,x) = b\overset{\circ 3}{\Longleftrightarrow}$
$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(a_1^{n-2}),A(a_1^{n-2},a,x)) = A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(a_1^{n-2}),b)$
$\overset{\circ 2,(\overline{2})}{\Longleftrightarrow} x = A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(a_1^{n-2}),b).$

c) By a) and $\circ 1$ and by b) and $\circ 3$, we obtain $\circ 4$.

Finally, by $\circ 2$, $\circ 4$ and by Prop. 2.2 from Chapter III, we conclude that (Q,A) is an $n-$group.

Remark: Similarly, it is possible to prove the case "$(1_R),(2_L),(2_R)$". See footnotes [6] and [7]. □

2.3. Theorem *[Ušan 1997/2]: Let* (Q,A) *be an* $n-$*groupoid and* $n \geq 3$. *Then there is* **at most one** *mapping* $\mathbf{e}$ *of the set* Q^{n-2} *into the set* Q *such that the laws* $(\overline{1}) - (\overline{3})$ *from Th. 2.2 hold in the algebra* $(Q,\{A,\mathbf{e}\})$ *of the type* $< n, n-2 >$.

Proof. By $(\overline{2})$, $(\overline{3})$, Def. 2.1 Chapter II and by Prop. 2.3 from Chapter II.

Remark: Similarly, it is possible to prove the case "$(1_L),(2_L),(2_R)$". □

2.4. Theorem *[Ušan 1997/2]: The laws* $(\overline{1}) - (\overline{3})$ *from Theorem 2.2 are mutually independent.*

Proof. a) The laws $(\overline{1})$ and $(\overline{2})$ from Th. 2.2 hold in the algebra $(Q,\{A,\mathbf{e}\})$

of the type $< n, n-2 >$, where $n \geq 3$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_n$, and $\mathbf{e}(a_1^{n-2}) \overset{def}{=}$ $c-$constant. However, the law $(\overline{3})$ from Th. 2.2 does not hold.

b) The laws $(\overline{1})$ and $(\overline{3})$ from Th. 2.2 hold in the algebra $(Q, \{A, \mathbf{e}\})$ of the type $< n, n-2 >$, where $n \geq 3$, $|Q| > 1$, $A(x_1^n) \overset{def}{=} x_1$, and $\mathbf{e}(a_1^{n-2}) \overset{def}{=}$ $c-$constant. However, the law $(\overline{2})$ from Th. 2.2 does not hold.

c) Let $(Q, \Box)$ be a group, $^{\mathbf{-1}}$ its inverse operation, $n \geq 3$, and let (Q, B) be an $(n-2)-$groupoid which is not an $(n-2)-$quasigroup. Then (Q, A), where

$$A(x, a_1^{n-2}, y) \overset{def}{=} x \Box (B(a_1^{n-2}))^{\mathbf{-1}} \Box y,$$

satisfies conditions of Proposition 2.5 from Chapter II. Thus, there is an algebra $(Q, \{A, \mathbf{e}\})$ of the type $< n, n-2 >$, in which the laws $(\overline{2})$ and $(\overline{3})$ from Th. 2.2 hold. Howeover, the law $(\overline{1})$ from Th. 2.2 fails to hold in $(Q, \{A, \mathbf{e}\})$. Indeed, if the law $(\overline{1})$ from Th. 2.2 hold in $(Q, \{A, \mathbf{e}\})$ then by Th. 2.2 is an $n-$group, which contradict the assumption that (Q, B) is not an $(n-2)-$quasigroup. $\Box$

2.5. Remark: *The part of Theorem 1.4 from the paper [Monk, Sioson 1971] is the following proposition:*

2.5.1. Let $n \geq 3$ and let (Q, A) be an $n-$semigroup. Then: (Q, A) is an $n-$group iff for every $a_1^{n-2} \in Q$ there is exactly one $(a_1, \ldots, a_{n-2})^{-1} \in Q$ such that for every $x \in Q$ the following equalities hold

$$A(x, a_1^{n-2}, (a_1, \ldots, a_{n-2})^{-1}) = x, \; A(a_1^{n-2}, (a_1, \ldots, a_{n-2})^{-1}, x) = x,$$
$$A((a_1, \ldots, a_{n-2})^{-1}, a_1^{n-2}, x) = x, \; A(x, (a_1, \ldots, a_{n-2})^{-1}, a_1^{n-2}) = x.$$

Operation $^{-1}$ is a $\{1, n\}-$neutral operation of the $n-$semigroup (Q, A).

Moreover, in [Celakoski 1977] the following proposition was shown:

2.5.2. An $n-$semigroup (Q, A) is an $n-$group $(n \geq 3)$ iff there exists an $(n-2)-$ary operation $^{-1}$ on Q such that for any $x_1, \ldots, x_{n-2}, y \in Q$ the following hold

$$A(y, x_1, \ldots, x_{n-2}, (x_1, \ldots, x_{n-2})^{-1}) = y = A((x_1, \ldots, x_{n-2})^{-1}, x_1, \ldots, x_{n-2}, y).$$

In Th. 2.1 as well as in Prop. 2.5.2, the $n-$semigroup (Q, A), $n \geq 3$, *is followed by* $(n-2)-$*ary operation in* Q. *In addition, if the fact that the mentioned* $(n-2)-$*ary operation in Th. 2.1 generalizes the notion of a neutral element of a groupoid [: Chapter II-2] is omited and in Prop. 2.5.2 its denotation [:* $^{-1}$ *] is ignored, then we can say that they represent the same proposition.*

3 Some more propositions

Note that the following proposition has been proved in [Tvermoes 1953]:

3.1. Theorem *[Post 1940* [8]*, Tvermoes 1953]: An* $n-$*semigroup* (Q, A) *is an* $n-$*group iff for each* $a_1^n \in Q$ *there exists* **at least one** $x \in Q$ *and* **at least one** $y \in Q$ *such that the following equalities hold:* $A(a_1^{n-1}, x) = a_n$ *and* $A(y, a_1^{n-1}) = a_n$.

This result was improved in [Ušan 1997/3]:

3.2. Theorem *[Ušan 1997/3]: Let* (Q, A) *be an* $n-$*groupoid and let* $n \geq 2$. *Then,* (Q, A) *is an* $n-$*group iff the following statements hold:*

(i) (Q, A) *is an* $< n-1, n > -$*associative* $n-$*groupoid* [9];

(ii) (Q, A) *is an* $< 1, n > -$*associative* $n-$*groupoid;*

(iii) *For every* $a_1^n \in Q$ *there is* **at least one** $x \in Q$ *such that the following equality* $A(a_1^{n-1}, x) = a_n$ *holds; and*

(iv) *For every* $a_1^n \in Q$ *there is* **at least one** $y \in Q$ *such that the following equality* $A(y, a_1^{n-1}) = a_n$ *holds.*

Proof. 1) $\Rightarrow$: By Def.1.1 from Chapter I.
2) $\Leftarrow$: Firstly we prove the following statement:

*1 There are mappings $^{-1}$ and **e**, respectively, of the sets Q^{n-1} and Q^{n-2}

[8] This assertion has been already formulated in [Post 1940], but the proof is missing there.

[9] or: $\widehat{(i)}$ $< 1, 2 >$-associative $n-$groupoid.

into the set Q such that for all $a, x \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ and
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$.

The proof of *1:

By $(ii) - (iv)$, by Prop. 2.5 from Chapter II and by Prop. 1.2 from Chapter III.

Finally, by (i), by *1 and by Th. 3.1 from Chapter III, we conclude that the proposition is satisfied.

Remark: Similarly, it is possible to prove the case "$\widehat{(i)}, (ii), (iv)$". See footnote [9]. □

3.3. Remark: *A group as a semigroup and a quasigroup was characterized by Weber H. in 1896 (cf. [Clifford, Preston 1964], p.p. 19-20). A notion of an $n-$group was introduced by Dörnte W. in [Dörnte 1928] as a generalization of Weber's characterization of a group. A group as a semigroup $(Q, \cdot)$ in which the following formula holds*

$(\forall a \in Q)(\forall b \in Q)(\exists x \in Q)(\exists y \in Q)(a \cdot x = b \wedge y \cdot a = b)$

was characterized by Hungtington E.V. in 1902 (cf. [Cliford, Preston 1964], p.p. 20). For $n \geq 3$ see also [Tyutin 1985] and [Gal'mak 2000].

3.4. Theorem *[Ušan 1999/7]: Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there is $i \in \{2, \ldots, n-1\}$ such that the following statements hold:*

(a) The $< i-1, i > -$associative law holds in (Q, A);

(b) The $< i, i+1 > -$associative law holds in (Q, A); and

(c) For every $a_1^n \in Q$ there is **exactly one** *$x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.*

Remark: $(c) \Leftrightarrow (c_1) \wedge (c_2)$, where

(c_1) For every $a_1^{n-1}, x, y \in Q$ the implication holds
$A(a_1^{i-1}, x, a_i^{n-1}) = A(a_1^{i-1}, y, a_i^{n-1}) \Rightarrow x = y$; and

(c_2) For every $a_1^n \in Q$ there is **at least one** $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

Proof. 1) $\Rightarrow$: By Def. 1.1 from Chapter I.

2) $\Leftarrow$: Firstly we prove the following statements:

$\overset{\circ}{1}$ (Q, A) is an $n-$semigroup;

$\overset{\circ}{2}$ For every $a_1^n \in Q$ there is at least one $x \in Q$ such that the following equality holds $A(a_1^{n-1}, x) = a_n$; and

$\overset{\circ}{3}$ For every $a_1^n \in Q$ there is at least one $y \in Q$ such that the following equality holds $A(y, a_1^{n-1}) = a_n$.

Sketch of the proof of $\overset{\circ}{1}$:

a)$i \leq j \leq n-2$:

$$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1}) = A(a_1^j, A(a_{j+1}^{j+n}), a_{j+n+1}^{2n-1}) \Rightarrow$$
$$A(b_1^i, A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1}), b_{i+1}^{n-1}) =$$
$$A(b_1^i, A(a_1^j, A(a_{j+1}^{j+n}), a_{j+n+1}^{2n-1}), b_{i+1}^{n-1}) \overset{(b)}{\Rightarrow}$$
$$A(b_1^{i-1}, A(b_i, a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-2}), a_{2n-1}, b_{i+1}^{n-1}) =$$
$$A(b_1^{i-1}, A(b_i, a_1^j, A(a_{j+1}^{j+n}), a_{j+n+1}^{2n-2}), a_{2n-1}, b_{i+1}^{n-1}) \overset{(c_1)}{\Rightarrow}$$
$$A(b_i, a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-2}) = A(b_i, a_1^j, A(a_{j+1}^{j+n}), a_{j+n+1}^{2n-2}).$$

b) $2 \leq k \leq i-1$:

$$A(a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}) = A(a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}) \Rightarrow$$
$$A(b_1^{i-2}, A(a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}), b_{i-1}^{n-1}) =$$
$$A(b_1^{i-2}, A(a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}), b_{i-1}^{n-1}) \overset{(a)}{\Rightarrow}$$
$$A(b_1^{i-2}, a_1, A(a_2^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}, b_{i-1}), b_i^{n-1}) =$$
$$A(b_1^{i-2}, a_1, A(a_2^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}, b_{i-1}), b_i^{n-1}) \overset{(c_1)}{\Rightarrow}$$
$$A(a_2^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}, b_{i-1}) = A(a_2^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}, b_{i-1}).$$

See, also the prof of Prop. 2.1 from Chapter III.

For $n = 3[i = 2]$ the statements (a) and (b) are equivalent to the statement that (Q, A) is a 3-semigroup.

Sketch of the proof of $\overset{\circ}{2}$:

$A(a_1^{n-1}, x) = a_n \overset{(c_1)}{\Leftrightarrow}$
$A(b_1^{i-1}, A(a_1^{n-1}, x), b_i^{n-1}) = A(b_1^{i-1}, a_n, b_1^{n-1}) \overset{\overset{\circ}{1}}{\Leftrightarrow}$
$A(b_1^{i-2}, A(b_{i-1}, a_1^{n-1}), x, b_i^{n-1}) = A(b_1^{i-1}, a_n, b_1^{n-1})$,
i.e. that
$A(a_1^{n-1}, x) = a_n \Leftrightarrow$
$A(b_1^{i-2}, A(b_{i-1}, a_1^{n-1}), x, b_i^{n-1}) = A(b_1^{i-1}, a_n, b_1^{n-1})$,
where b_1^{n-1} is an arbitrary sequence over Q.
Whence, by (c), we conclude that the statement $\overset{\circ}{2}$ holds.

Similarly, it is possible to prove the statement $\overset{\circ}{3}$.

Finally, by $\overset{\circ}{1} - \overset{\circ}{3}$ and by Th. 3.1 (3.2), we conclude that the proposition is satisfied. □

3.5. Remark: *A part of 1.4 in [Monk, Sioson 1971] is the following proposition. Let $n \geq 3$ and let (Q, A) be an $n-$semigroup. Then (Q, A) is an $n-$group iff for some $i \in \{2, \ldots, n-1\}$ and for every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$. See, also [Tyutin 1985].*

Chapter X

CENTRAL OPERATIONS ON $n-$GROUPS

1 Notion and main propositions

1.1. Definition *[Ušan 2001/1]: Let (Q,A) be an $n-$group and $n \geq 2$. Also, let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in the set Q. We say that $\boldsymbol{\alpha}$ is a* **central operation** *of the $n-$group (Q,A) iff for every sequence a_1^{n-2} over Q, for every sequence b_1^{n-2} over Q and for every $x \in Q$ the following equality holds*

(1) $$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}).$$

1.2. Remarks: *a) If $n = 2$, then $\boldsymbol{\alpha}(c_1^{n-2})[= \boldsymbol{\alpha}(c_1^{\circ}) = \boldsymbol{\alpha}(\emptyset) = c \in Q]$ is a* **central element of the group** *(Q,A); and b) The $\{1,n\}-$neutral operation $\mathbf{e}$ of the $n-$group (Q,A) is a central operation of that $n-$group (cf. Proposition 1.1 from Chapter IV).*

1.3. Proposition *[Ušan 2001/1]: Let (Q,A) be an $n-$group and $n \geq 2$. Also, let $\boldsymbol{\alpha}$ be a central operation of the $n-$group (Q,A). Then for every $i \in \{1,\ldots,n\}$, for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds*

(a) $$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, A(x_1^n)) = A(x_1^{i-1}, A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x_i), x_{i+1}^n).$$

Proof. 1) $n = 2$: For $n = 2$ $\boldsymbol{\alpha}(c_1^{n-2})[= \boldsymbol{\alpha}(\emptyset)]$ is an element of the center of the group (Q,A).

2) $n \geq 3$: a) Since, by the assumption, (Q,A) is an $n-$group, we conclude that for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold

$$\begin{aligned} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, A(x_1^n)) &= A(A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_1), x_2^n) \\ &= A(A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x_1), x_2^n). \end{aligned}$$

b) Since, by the assumption, (Q,A) is an $n-$group and $\boldsymbol{\alpha}$ its central operation, we conclude that for every $j \in \{1,\dots,n-1\}$, for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold

$$\begin{aligned} A(x_1^{j-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_j), x_{j+1}^n) &\overset{(1)}{=} A(x_1^{j-1}, A(x_j, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}), x_{j+1}^n) \\ &= A(x_1^j, A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x_{j+1}), x_{j+2}^n). \end{aligned}$$

□

1.4. Proposition *[Ušan 2001/1]: Let (Q,A) be an $n-$group and $n \geq 2$. Also, let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in the set Q. Further on, let for all $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds*

(2) $$A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x).$$

Then for every $i \in \{1,\dots,n\}$, for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

(b) $$A(A(x_1^n), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(x_1^{i-1}, A(x_i, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_{i+1}^n).$$

Sketch of the proof. $\overline{1}$) $n=2$: For $n=2$ $\boldsymbol{\alpha}(c_1^{n-2}[=\boldsymbol{\alpha}(\emptyset)]$ is an element of the center of the group (Q,A).

$\overline{2}$) $n \geq 3$:

$\overline{a}$) $$\begin{aligned} A(A(x_1^n), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) &= A(x_1^{n-1}, A(x_n, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))) \\ &= A(x_1^{n-1}, A(x_n, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}))). \end{aligned}$$

$\overline{b}$) $$\begin{aligned} A(x_1^j, A(x_{j+1}, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x_{j+2}^n) &\overset{(2)}{=} A(x_1^j, A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x_{j+1}), x_{j+2}^n) \\ &= A(x_1^{j-1}, A(x_j, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_{j+1}^n). \end{aligned}$$

□

1.5. Proposition *[Ušan 2001/1]: Let $n \geq 2$, (Q,A) be an $n-$group and $\boldsymbol{\alpha}$ its central operation. Then, for every sequence a_1^{n-2} over Q, for every sequence b_1^{n-2} over Q, for every sequence c_1^{n-2} over Q, for every sequence d_1^{n-2} over Q and for every $x \in Q$ the following equalities hold*

(3) $$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})) \text{ and}$$

(4) $$A(x, \boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}) = A(d_1^{n-2}, \boldsymbol{\alpha}(d_1^{n-2}), x).$$

Proof. 1) $n=2$: For $n=2$ the proposition is trivial.

2) $n \geq 3$:

Sketch of the proof of (3) :

$F(a_1^{n-2},x)\overset{def}{=}A(x,a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}))\Rightarrow$

$A(F(a_1^{n-2},x),a_1^{n-2},z)=A(A(x,a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2})),a_1^{n-2},z)\Rightarrow$

$A(F(a_1^{n-2},x),a_1^{n-2},z)=A(x,a_1^{n-2},A(\boldsymbol{\alpha}(a_1^{n-2}),a_1^{n-2},z))\overset{1.3}{\Rightarrow}$

$A(F(a_1^{n-2},x),a_1^{n-2},z)=A(A(\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2},x),a_1^{n-2},z)\Rightarrow$

$F(a_1^{n-2},x)=A(\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2},x).$

Sketch of the proof of (4) :

$\Phi(a_1^{n-2},x)\overset{def}{\Rightarrow}A(a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}),x)\Rightarrow$

$A(a_1^{n-2},z,\Phi(a_1^{n-2},x))=A(a_1^{n-2},z,A(a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}),x))\Rightarrow$

$A(a_1^{n-2},z,\Phi(a_1^{n-2},x))=A(a_1^{n-2},A(z,a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2})),x)\overset{(3)}{\Rightarrow}$

$A(a_1^{n-2},z,\Phi(a_1^{n-2},x))=A(a_1^{n-2},A(\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2},z),x)\overset{1.3}{\Rightarrow}$

$A(a_1^{n-2},z,\Phi(a_1^{n-2},x))=A(a_1^{n-2},z,A(\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2},x))\overset{(1)}{\Rightarrow}$

$A(a_1^{n-2},z,\Phi(a_1^{n-2},x))=A(a_1^{n-2},z,A(x,\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2}))\Rightarrow$

$\Phi(a_1^{n-2},x)=A(x,\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2}).$ □

Similarly, it is possible to prove that the following proposition holds:

1.6. Proposition *[Ušan 2001/1]: Let (Q,A) be an $n-$group and $n\geq 2$. Also, let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in the set Q. Further on, let for all $x\in Q$, for every sequence a_1^{n-2} over Q, and for every sequence b_1^{n-2} over Q the equality (2) holds. Then, for every sequence a_1^{n-2} over Q, for every sequence b_1^{n-2} over Q, for every sequence c_1^{n-2} over Q, for every sequence d_1^{n-2} over Q and for every $x\in Q$ the equalities (3) and (4) hold.*

Sketch of a part of the proof.

$F(a_1^{n-2},x)\overset{def}{=}A(\boldsymbol{\alpha}(a_1^{n-2}),a_1^{n-2},x)\Rightarrow$

$A(z, a_1^{n-2}, F(a_1^{n-2}, x)) = A(z, a_1^{n-2}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x)) \Rightarrow$

$A(z, a_1^{n-2}, F(a_1^{n-2}, x)) = A(A(z, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, x) \overset{1.4}{\Rightarrow}$

$A(z, a_1^{n-2}, F(a_1^{n-2}, x)) = A(z, a_1^{n-2}, A(x, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}))) \Rightarrow$

$F(a_1^{n-2}, x) = A(x, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}))$. □

1.7. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group and $n \geq 2$. Also, let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in the set Q. Then the following statements are equivalent:*

(i) $\boldsymbol{\alpha}$ is a central operation of the $n-$group (Q, A); and

(ii) For all $x \in Q$, for every sequence a_1^{n-2} over Q, and for every sequence b_1^{n-2} over Q the equality (2) holds.

Sketch of the proof.

1) $(i) \Rightarrow (ii)$:
By Proposition 1.5:

$$\begin{array}{c} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})) \\ \| \\ A(x, \boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}) = A(d_1^{n-2}, \boldsymbol{\alpha}(d_1^{n-2}), x). \end{array}$$

2) $(ii) \Rightarrow (i)$:
By Proposition 1.6:

$$\begin{array}{c} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})) \\ \| \\ A(x, \boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}) = A(d_1^{n-2}, \boldsymbol{\alpha}(d_1^{n-2}), x). \ \square \end{array}$$

A direct consequence of Proposition 1.5 [:(3)] is the following proposition:

1.8. Proposition: *Let (Q, A) be an $n-$group and $n \geq 2$. Also, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the $n-$group (Q, A). Then for every sequence a_1^{n-2} over Q, the following equality holds*

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\beta}(a_1^{n-2})) = A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})).$$

1.9. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $\boldsymbol{\alpha}$ its central* *operation and $n \geq 2$. Then there is a permutation α of the set Q such that for every $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following conjuction of equalities holds*

(5) $$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = \alpha(x) \wedge A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}) = \alpha(x).$$

Proof. Let k_1^{n-2} be an arbitrary chosen sequence over the set Q. Then, α defined by

$$\alpha(x) \overset{def}{=} A(x, \boldsymbol{\alpha}(k_1^{n-2}), k_1^{n-2})$$

for every $x \in Q$, is a permutation of the set Q, since (Q, A) is an $n-$quasigroup. Hence, by Def. 1.1, we conclude that the proposition holds. (For $n = 2$: $\boldsymbol{\alpha}(k_1^{n-2}) = \boldsymbol{\alpha}(\emptyset) \in Q$.) □

1.10. Definition *Let (Q, A) be an $n-$group, $\boldsymbol{\alpha}$ its central operation and $n \geq 2$. Also, let α be a permutation of the set Q. We shall say that α is* **associated** *to $\boldsymbol{\alpha}$ iff for all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds*

$$\alpha(x) = A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x).$$

1.11. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $\boldsymbol{\alpha}$ its central operation and $n \geq 2$. Also, let α be associated to $\boldsymbol{\alpha}$. Then for every $i \in \{1, \ldots, n\}$ and for every $x_1^n \in Q$ the following equality holds:*

$$\alpha A(x_1^n) = A(x_1^{i-1} \alpha(x_i), x_{i+1}^n).$$

Proof. By Prop. 1.3, Theorem 1.9 and Def. 1.10. □

1.12. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $\boldsymbol{\alpha}[\boldsymbol{\beta}]$ its central operation and $n \geq 2$. Also, let $\alpha[\beta]$ be associated to $\boldsymbol{\alpha}[\boldsymbol{\beta}]$. Then for every $x \in Q$ the following equality holds*

$$\alpha(\beta(x)) = \beta(\alpha(x)).$$

Sketch of the proof.

$$\begin{aligned}\alpha(\beta(x)) &\overset{1.11}{=} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \beta(x))\\ &\overset{1.11}{=} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x))\\ &= A(A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\beta}(a_1^{n-2})), a_1^{n-2}, x)\\ &\overset{1.8}{=} A(A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, x)\\ &= A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x))\\ &= A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \alpha(x))\\ &= \beta(\alpha(x)). \quad \square\end{aligned}$$

1.13. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $\boldsymbol{\alpha}$ its central operation, $^{-1}$ its inverse operation (cf. 1.3. from Chapter III) and $n \geq 2$. Also, let α be associated to $\boldsymbol{\alpha}$, and let for all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds*

(a) $\quad (a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1} = \boldsymbol{\alpha}(a_1^{n-2}).$

Then, for all $x \in Q$ the following equality holds:

$$\alpha(\alpha(x)) = x.$$

Sketch of the proof.

$$\begin{aligned}\alpha(\alpha(x)) &\overset{1.10}{=} A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x))\\ &= A(A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, x)\\ &\overset{(a)}{=} A(A((a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1}, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, x)\\ &\overset{1.3\,III}{=\!=} A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) \overset{1.3\,III}{=\!=} x.\end{aligned}$$

(1.3 III: Theorem 1.3 from Chapter III.) $\square$

1.14. Examples: *Let $(\{1, 2, 3, 4\}, \cdot)$ be the Klein group: [Tab. 1] and $^{-1}$ the corresponding inverse operation. Further on, let φ be the permutation of the set $\{1, 2, 3, 4\}$ defined in the following way*

$$\varphi \overset{def}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Tab. 1

Then, $\varphi \in Aut(Q, \cdot), (\forall x \in \{1, 2, 3, 4\}\varphi^2(x) = x,\ \varphi(2) = 2$ and $\varphi(1) = 1$.

1.14a Example: *Let $A(x_1^3) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot x_3 \cdot 2$ and $\boldsymbol{\alpha}(c) \overset{def}{=} 3 \cdot (\varphi(c))^{-1}$ for every $x_1^3, c \in \{1, 2, 3, 4\}$. Then: (i) $(\{1, 2, 3, 4\}, A)$ is a 3-group [cf. 1.4 from Chapter I]; and (ii) for every $c \in \{1, 2, 3, 4\}$ the following equalities hold*

$A(\boldsymbol{\alpha}(c), c, x) = 4x$, $A(x, c, \boldsymbol{\alpha}(c)) = 4x$ and $A(x, \boldsymbol{\alpha}(c), c) = 3x$ [cf. Prop. 1.5].

1.14b Example: *Let $B(x_1^3) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot x_3$ and $\boldsymbol{\beta}(c) \overset{def}{=} 2 \cdot (\varphi(c))^{-1}$ for every $x_1^3, c \in \{1, 2, 3, 4\}$. Then: (a) $(\{1, 2, 3, 4\}, B)$ is a 3-group; and (b) for every $c \in \{1, 2, 3, 4\}$ the following equalities hold $B(\boldsymbol{\beta}(c), c, x) = 2x$ and $B(x, \boldsymbol{\beta}(c), c) = 2x$.* □

2 A description of central operations in terms of Hosszú - Gluskin algebras

2.1 Lemma *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $n \geq 3$, and let $(Q, \{\cdot, \varphi, b\})$ be an $nHG-$algebra associated to the $n-$group (Q, A). Also, let $\boldsymbol{\alpha}$ be a central operation on the $n-$group (Q, A). Then there is exactly one constant $a \in Q$ such that for every sequence a_1^{n-2} over Q, the equality*

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) = a$$

holds.

Proof. Let c_1^{n-2} be an arbitrary [fixed], sequence over Q. Then, by 1.9, for every $x \in Q$ and for every sequence a_1^{n-2} over Q the equality

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(\boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}, x)$$

holds, from which, by Hosszú - Gluskin Theorem [Chapter IV], we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over$Q$ the equality

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b \cdot x = \boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2}) \cdot b \cdot x$$

holds, i.e.,

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) = \boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2})$$

holds. Hence, since by the assumption, c_1^{n-2} is a fixed sequence over Q, by the convention the constant $\boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2})$ is denoted by a, we conclude that for every sequence a_1^{n-2} over Q the following equality holds

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) = a. \quad \square$$

2.2. Theorem *[Ušan 2001/1]: Let $n \geq 3$, (Q, A) an $n-$group, $(Q, \{\cdot, \varphi, b\})$*

its associated $nHG-$algebra [Chapter IV], and $^{\mathbf{-1}}$ *the inverse operation in the group $(Q,\cdot)$. Also, let $\boldsymbol{\alpha}$ be a central operation of the $n-$group (Q,A) and let the permutation α be associated to $\boldsymbol{\alpha}$. Then there is exactly one constant $a\in Q$ such that for every $x\in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

(1) $\boldsymbol{\alpha}(a_1^{n-2})=a\cdot(\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))^{\mathbf{-1}}$,

(2) $\alpha(x)=(a\cdot b)\cdot x$,

(3) $\varphi(a)=a$ *and*

(4) $(a\cdot b)\cdot x=x\cdot(a\cdot b)$.

Proof. 1) Since $(Q,\cdot)$ is a group and $^{\mathbf{-1}}$ its inverse operation, by Lemma 2.1, we conclude that there is exactly one constant $a\in Q$ such that for every sequence a_1^{n-2} over Q the equality (1) holds.

2) By the assumption of proposition, by Hosszú - Gluskin Theorem *[*Chapter IV *]*, we conclude that for every $x\in Q$ and for every a_1^{n-2} over Q the equality

$$\alpha(x)=\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot x$$

holds, and from there, by Lemma 2.1, we conclude that there is exactly one constant $a\in Q$ such that for every $x\in Q$ equality (2) holds.

3) Considering Def.1.1, Hosszú - Gluskin Theorem and by the fact $\varphi\in Aut(Q,\cdot)$ *[*Chapter IV *]*, we conclude that for every $x\in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the equality

$\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot x=x\cdot\varphi(\boldsymbol{\alpha}(b_1^{n-2})\cdot\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2})\cdot b$

holds, and from there, by Lemma 2.1, we conclude that there is exactly one constant $a\in Q$ such that for every $x\in Q$ the equality

$(\overline{4})$ $a\cdot b\cdot x=x\cdot\varphi(a)\cdot b$

holds. Putting $x=e$ in $(\overline{4})$, where e is a neutral element of the group $(Q,\cdot)$, we obtain (3).

4) Putting (3) in $(\overline{4})$ we obtain (4). □

2.3. Theorem *[Ušan 2001/1]: Let $n\geq 3$, (Q,A) an $n-$group, $(Q,\{\cdot,\varphi,b\})$ its associated $nHG-$algebra [Chapter IV],* $^{\mathbf{-1}}$ *the inverse operation in the* *group* $(Q,\cdot)$ *and* e *the neutral operation of the group* $(Q,\cdot)$*. Also, let* $\boldsymbol{\alpha}$ *be a central operation of the* $n-$*group* (Q,A) *and let the permutation* α *be associated to* $\boldsymbol{\alpha}$*. Further on, let for every sequence* a_1^{n-2} *the following equality holds*

$$(a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}))^{-1}=\boldsymbol{\alpha}(a_1^{n-2}),$$

where $^{-1}$ *is an inverse operation in the* $n-$*group* (Q,A)*. Then there is exactly one constant* $a\in Q$ *such that for all* $x\in Q$ *the following equality holds*

(5) $\qquad (a\cdot b)\cdot(a\cdot b)=e.$

Sketch of the proof.

$(a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}))^{-1}=\boldsymbol{\alpha}(a_1^{n-2})\overset{1.3III}{\Longrightarrow}$
$A(\boldsymbol{\alpha}(a_1^{n-2}),a_1^{n-2},\boldsymbol{\alpha}(a_1^{n-2}))=\mathbf{e}(a_1^{n-2})\overset{4.2IV}{\Longrightarrow}$
$\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot\boldsymbol{\alpha}(a_1^{n-2})=(\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b)^{-\mathbf{1}}\Rightarrow$
$(\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))\cdot b\cdot(\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))\cdot b=e\overset{2.1}{\Longrightarrow}$
$(a\cdot b)\cdot(a\cdot b)=e.\quad\square$

2.4. Theorem *[Ušan 2001/1]: Let* $n\geq 3$*,* (Q,A) *be* $n-$*group,* $(Q,\{\cdot,\varphi,b\})$ *its associated* $nHG-$*algebra and* $^{-\mathbf{1}}$ *the inverse operation in the group* $(Q,\cdot)$*. Also, let* a *be a fixed element of the set* Q *such that for all* $x\in Q$ *the following equalities hold*

(a) $\qquad (a\cdot b)\cdot x=x\cdot(a\cdot b)$ *and*

(b) $\qquad \varphi(a)=a.$

Further on, let

(c) $\qquad \boldsymbol{\alpha}(a_1^{n-2})\overset{def}{=}a\cdot(\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))^{-\mathbf{1}}$

for every sequence a_1^{n-2} *over* Q*. Then* $\boldsymbol{\alpha}$ *is a central operation on the* $n-$*group* (Q,A)*.*

Sketch of the proof.

$$\begin{aligned}
A(\boldsymbol{\alpha}(a_1^{n-2}),a_1^{n-2},x)&\overset{3.1IV}{=}\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot x\\
&\overset{(c)}{=}a\cdot(\varphi(a_1^{n-2})\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))^{-\mathbf{1}}\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot x,\\
&=a\cdot b\cdot x,\\
A(x,\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2})&\overset{3.1IV}{=}x\cdot\varphi(\boldsymbol{\alpha}(b_1^{n-2}))\cdot\varphi(\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2}))\cdot b\\
&\overset{(c)}{=}x\cdot\varphi(a\cdot(\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2}))^{-\mathbf{1}})\cdot\varphi(\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2}))\cdot b\\
&\overset{2IV}{=}x\cdot\varphi(a)\cdot\varphi((\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2}))^{-\mathbf{1}})\cdot\varphi(\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(b_{n-2}))\cdot b
\end{aligned}$$

$$\overset{(b)}{=} x \cdot a \cdot b$$
$$\overset{(a)}{=} a \cdot b \cdot x. \quad \square$$

2.5. Theorem *[Ušan 2001/1]: Let $n \geq 3$, (Q, A) be an $n-$group, $(Q, \{\cdot, \varphi, b\})$ its associated $nHG-$algebra, $^{\mathbf{-1}}$ the inverse operation in the group $(Q, \cdot)$ and e the neutral element of the group $(Q, \cdot)$. Also, let a be a fixed element of the set Q such that for all $x \in Q$ the following equalities hold*

(i) $\quad (a \cdot b) \cdot x = x \cdot (a \cdot b),$

(ii) $\quad \varphi(a) = a$ *and*

(iii) $\quad (a \cdot b) \cdot (a \cdot b) = e.$

Further on, let

(iv) $\quad \boldsymbol{\alpha}(a_1^{n-2}) \overset{def}{=} a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{\mathbf{-1}}$

for every sequence a_1^{n-2} over Q. Then: 1° *$\boldsymbol{\alpha}$ is a central operation on the $n-$group (Q, A); and* 2° *for every sequence a_1^{n-2} over Q the following equality holds*

$$(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1} = \boldsymbol{\alpha}(a_1^{n-2}),$$

where $^{-1}$ is an inverse operation in the $n-$group (Q, A).

Proof. The proof of 1° : The proof of Th. 2.4.

Sketch of the proof of 2° :

1) By Th. 3.1 from Chapter III and by Prop. 4.2 from Chapter IV, we conclude that for all $x, a_1^{n-2} \in Q$ the following equality holds

(v) $(a_1^{n-2}, x)^{-1} = (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b)^{\mathbf{-1}} \cdot x^{\mathbf{-1}} \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b)^{\mathbf{-1}}$

2) $(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1}$

$\overset{(v)}{=} (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b)^{\mathbf{-1}} (\boldsymbol{\alpha}(a_1^{n-2}))^{\mathbf{-1}} \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b)^{\mathbf{-1}}$

$= ((\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b) \cdot \boldsymbol{\alpha}(a_1^{n-2}) \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b))^{\mathbf{-1}}$

$\overset{(iv)}{=} ((\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b) \cdot a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{\mathbf{-1}} \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b))^{\mathbf{-1}}$

$= ((\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b \cdot a \cdot b))^{\mathbf{-1}}$

$= b^{\mathbf{-1}} \cdot a^{\mathbf{-1}} \cdot b^{\mathbf{-1}} \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{\mathbf{-1}}$

$= b^{\mathbf{-1}} \cdot a^{\mathbf{-1}} \cdot b^{\mathbf{-1}} \cdot a^{\mathbf{-1}} \cdot a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{\mathbf{-1}}$

$\stackrel{(iv)}{=} (a \cdot b)^{\mathbf{-1}} \cdot (a \cdot b)^{\mathbf{-1}} \cdot \boldsymbol{\alpha}(a_1^{n-2})$
$= ((a \cdot b) \cdot (a \cdot b))^{\mathbf{-1}} \cdot \boldsymbol{\alpha}(a_1^{n-2})$
$\stackrel{(iii)}{=} \boldsymbol{\alpha}(a_1^{n-2}). \quad \square$

Chapter XI

SUPER–ASSOCIATIVE ALGEBRAS WITH $n-$QUASIGROUP OPERATIONS

1 Introduction

Let $x_1, \ldots, x_{2n-1}$ be **subject symbols**, $n \in N \backslash \{1\}$, and let $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \ldots, n\}$ be $n-$**ary operational symbols**. Then, we say that
(1) $\qquad X_1(X_2(x_1^n), x_{n+1}^{2n-1}) = X_{2i-1}(x_1^{i-1}, X_{2i}(x_i^{i+n-1}), x_{i+n}^{2n-1})$
is a **general** $< 1, i >$ **−associative law**.

Some of operational symbols in (1) can be equal. In the case all of them are mutually equal, the (ordinary) $< 1, i > -$ associative law is in question.

For example, each of the following laws is a **general** $< 1, 2 >$ **−associative** (associative) laws:
(a) $\qquad X_1(X_2(x, y), z) = X_2(x, X_1(y, z))$,
(b) $\qquad X_1(X_2(x, y), z) = X_1(x, X_2(y, z))$ and
(c) $\qquad X_1(X_1(x, y), z) = X_2(x, X_2(y, z))$.

1.1. Definition *[Ušan 2001/1] Let $(Q, \sum)$ be an algebra in which the following holds: (Q, Z) is an $n-$quasigroup for every $Z \in \sum$. Also let $n \geq 2$ and $|\sum| \geq 2$. Further on, let $x_1, \ldots, x_{2n-1}$ be subject symbols, let $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \ldots, n\}$, be $n-$ary operational symbols, and let for all $i \in \{2, \ldots, n\}$ is $|\{X_1, X_2, X_{2i-1}, X_{2i}\}| \geq 2$. Then, we say that $(Q, \sum)$ is a* **super-associative algebra with** $n-$**quasigroup operations** *(briefly: SAAnQ) iff for every substitution of the subject symbols $x_1, \ldots, x_{2n-1}$ in (1) by elements $\overline{x}_1, \ldots, \overline{x}_{2n-1}$ of Q and for every substitution of the operational symbols $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \ldots, n\}$, in (1) by elements*

$\overline{X}_1, \overline{X}_2, \overline{X}_{2i-1}, \overline{X}_{2i},\ i \in \{2, \ldots, n\}$, *of* $\sum$ *for all* $i \in \{2, \ldots, n\}$ *the following equality holds:*

$(\overline{1})$ $\qquad \overline{X}_1(\overline{X}_2(\overline{x}_1^n), \overline{x}_{n+1}^{2n-1}) = \overline{X}_{2i-1}(\overline{x}_1^{i-1}, \overline{X}_{2i}(\overline{x}_i^{i+n-1}), \overline{x}_{i+n}^{2n-1}).$

(In [Ušan 2001/1]: nontrivial super associative algebra with $n-$quasigroup operations. See, also [Belousov 1965], p.86.)

A immediate consequence of Def.1.1 and Def.1.1 from Chapter I, is the following proposition:

1.2. Proposition If $(Q, \sum)$ is a $SAAnQ$, $n \in N\backslash\{1\}$, then (Q, Z) is an $n-$group for every $Z \in \sum$.

Case $n = 2$ is described in [Belousov 1965].

1.3. Theorem *[Belousov 1965]: Let $(Q, \sum)$ be a $SAA2Q$, and let $\cdot$ be an arbitrary element of $\sum$. Then the following statements hold:*

(i) (Q, Z) *is an group for every* $Z \in \sum$;

$(ii) |\{X_1, X_2, X_3, X_4\}| = 2$ *and* $(1) = (a)$ *or* $(1) = (b)$ *or* $(1) = (c)$;

(iii) *For every* $A \in \sum$ *there is exactly one* $a \in Q$ *such that for every* $x, y \in Q$ *the equality*

$$A(x, y) = x \cdot a \cdot y$$

holds;

(iv) *If* $(1) = (b)$, *then* $a \in C$ *(a from* (iii)*), where* C *is the center of the group* $(Q, \cdot)$;

(v) *If* $(1) = (c)$, *then* $a \in C$ *and* $a^{-1} = a$ *(a from* (iii)*), where* C *is the center of the group* $(Q, \{\cdot, ^{-1}\})$; *and*

(vi) *If* $(1) = (a)$ *and* $(1) \neq (b)$, *then* $a \in Q\backslash C$ *(a from* (iii)*), where* C *is the center of the group* $(Q, \cdot)$.

1.4. Remarks: *a) Case $n = 3$ Yu. Movsisyan was described in 1984 (cf. [Movsisyan 1986]). b) Case $n \geq 3$ was described in [Ušan 2001/1].*

2 The types of $SAAnQ$

2.1. Theorem: *Let $(Q, \sum)$ be an $SAAnQ$ and $n \in N \backslash \{1\}$. Then the following statements hold:*

1° $X_1 \neq X_2 \Rightarrow \{X_{2i-1}, X_{2i}\} = \{X_1, X_2\}$ *and*

2° $X_1 = X_2 \Rightarrow X_{2i-1} = X_{2i}$

for all $i \in \{2, \ldots, n\}$, where $X_1, X_2, X_{2i-1}, X_{2i}$ from 1.1-(1).

Proof. 1) Case $X_1 \neq X_2$.

a) Let $A, B, C, D, \overline{D}$ be arbitrary operations from $\sum, a_1, \ldots, a_{2n-1}$ arbitrary elements from Q and i an arbitrary element of the set $\{2, \ldots, n\}$ so that

$$A(B(a_1^n), a_{n+1}^{2n-1}) = C(a_1^{i-1}, D(a_i^{i+n-1}), a_{i+n}^{2n-1}) \text{ and}$$
$$A(B(a_1^n), a_{n+1}^{2n+1}) = C(a_1^{i-1}, \overline{D}(a_i^{i+n-1}), a_{i+n}^{2n-1}).$$

Since $\sum$ is a set of $n-$quasigroup operations, we conclude that $D = \overline{D}$, i.e. that X_{2i} from $2.1-(1)$ has no free choice for the substitution with operations from the set $\sum$. Similarly, we conclude that X_{2i-1} from $2.1-(1)$ has no free choice for the substitution with operations from the set $\sum$. Hence, for every $i \in \{2, \ldots, n\}$ it is true that

$$|\{X_1, X_2, X_{2i-1}, X_{2i}\}| < 4.$$

b) Let $A, B, C, \overline{C}$ be arbitrary elements from the set $\sum, a_1, \ldots, a_{2n-1}$ arbitrary elements from the set Q and i an arbitrary element from the set $\{2, \ldots, n\}$ so that

$$A(B(a_1^n), a_{n+1}^{2n-1}) = A(a_1^{i-1}, C(a_i^{i+n-1}), a_{i+n}^{2n-1}) \text{ and}$$
$$A(B(a_1^n), a_{n+1}^{2n-1}) = A(a_1^{i-1}, \overline{C}(a_i^{i+n-1}), a_{i+n}^{2n-1}); \text{ or}$$
$$A(B(a_1^n), a_{n+1}^{2n-1}) = C(a_1^{i-1}, B(a_i^{i+n-1}), a_{i+n}^{2n-1}) \text{ and}$$
$$A(B(a_1^n), a_{n+1}^{2n-1}) = \overline{C}(a_1^{i-1}, B(a_i^{i+n-1}), a_{i+n}^{2n-1}).$$

Hence, since $\sum$ is a set of $n-$quasigroup operations, we conclude that with $X_1 = X_{2i-1}$ or $X_2 = X_{2i}$

$$|\{X_1, X_2, X_{2i-1}, X_{2i}\}| < 3$$

for every $i \in \{2, \ldots, n\}$. In the same way we conclude that with $X_1 = X_{2i}$ or

$X_2 = X_{2i-1}$ it is true that

$$|\{X_1, X_2, X_{2i-1}, X_{2i}\}| < 3$$

for every $i \in \{2, \ldots, n\}$.

c) Let A, B be arbitrary elements of the set $\sum$, $x_1, \ldots, x_{2n-1}$ arbitrary elements from Q, and i an arbitrary element of the set $\{2, \ldots, n\}$ so that

$$A(B(a_1^n), a_{n+1}^{2n-1}) = A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}) \text{ or}$$
$$A(B(a_1^n), a_{n+1}^{2n-1}) = B(a_1^{i-1}, B(a_i^{i+n-1}), a_{i+n}^{2n-1})$$

(cf. b)). Hence, since (Q, A) and (Q, B) are $n-$groups (Prop. 2.3), it follows that $A = B$, i.e. that X_1 or X_2 from 2.1-(1) has no free choice for the substitution with operations from $\sum$, which is a contradiction with the assumption that the following equality holds $|\{X_1, X_2\}| = 2$.

2) Case $X_1 = X_2$.

Let $A, B, C, \overline{C}$ be arbitrary elements from $\sum$, $a_1, \ldots, a_{2n-1}$ arbitrary elements from Q, and i an arbitrary element of the set $\{2, \ldots, n\}$ so that

$$A(A(a_1^n), a_{n+1}^{2n-1}) = B(a_1^{i-1}, C(a_i^{i+n-1}), a_{i+n}^{2n-1}) \text{ and}$$
$$A(A(a_1^n), a_{n+1}^{2n-1}) = B(a_1^{i-1}, \overline{C}(a_i^{i+n-1}), a_{i+n}^{2n-1}).$$

Hence, since $\sum$ is a set of $n-$quasigroup operations, we conclude that X_{2i} from 2.1-(1) has no free choice for the substitutions with operations in $\sum$. Similarly, we conclude that X_{2i-1} from 2.1-(1) also has no free choice for the substitution with operations from $\sum$. Thus, for every $i \in \{2, \ldots, n\}$, it is true that

$$|\{X_{2i-1}, X_{2i}\}| = 1. \ \square$$

2.2. Definition: *We will say that a $SAAnQ$ has type $XY(XX)$ iff* $1°(2°)$ *of Theorem 2.1 holds.*
(In [Ušan 2001/1] types XX and XY are denoted differently.)

3 A description of an $SAAnQ$ of the type XX

3.1. Theorem *[Ušan 2001/1]: Let $(Q, \sum)$ be an $SAAnQ$ of the type XX and let $n \geq 3$. Also, let A be an arbitrary operation from $\sum$ [cf. Prop. 1.2]. Then, for all $B \in \sum$ there is a central operation $\boldsymbol{\alpha}$ of the $n-$group (Q, A) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold:*

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$ *and*

$(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1} = \boldsymbol{\alpha}(a_1^{n-2})$,

where $^{-1}$ is an inverse operation in the $n-$group (Q, A).

Proof. Let A and B be two arbitrary operations from $\sum$. By Proposition 1.2, (Q, A) and (Q, B) are $n-$groups. By Theorem 2.6 from Chapter II, (Q, A) and (Q, B) have $\{1, n\}-$neutral operations, denoted, respectively, by $\mathbf{e}$ and $\mathbf{e}_B$. Let also the inverse operation in (Q, A) be denoted $^{-1}$ (cf. Chapter III).

The following statements hold:

1° For all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

(1) $\quad B(x_1^n) = A(x_1^{n-1}, A(x_n, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})))$;

2° For all $x \in Q$ and for every sequences a_1^{n-2} and b_1^{n-2} over Q the following equality holds

$$A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}), x) = A(x, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2})),$$

i.e. $\mathbf{e}_B$ is a central operation of the $n-$group (Q, A); and

3° For every sequence a_1^{n-2} over Q the following equality holds

$$(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1} = \mathbf{e}_B(a_1^{n-2}).$$

The proof of 1° :

By Def. 2.2 and by Prop.1.2 for every $x_1^{2n-1} \in Q$ the following equality holds

$$B(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1})),$$

hence, by the substitutions $x_{n+1}^{2n-2} = a_1^{n-2}$ and $x_{2n-1} = \mathbf{e}_B(a_1^{n-2})$, where a_1^{n-2}

is an arbitrary sequence over Q, we conclude that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the equality (1) holds.

The proof of 2° :

Since (Q,B) is an $n-$group, for every $x_1^{2n-1} \in Q$ the following equality holds

$$B(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1}),$$

hence, by the statement 1°, we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

$$A(A(x_1^{n-1}, A(x_n, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))), x_{n+1}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) =$$

$$A(x_1, A(x_2^n, A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2}))), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) \Rightarrow$$

$$A(x_1, A(x_2^{n-1}, A(x_n, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), x_{n+1}), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) =$$

$$A(x_1, A(x_2^n, A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2})), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) \Rightarrow$$

$$A(x_1, A(x_2^{n-1}, x_n, A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}), x_{n+1})), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) =$$

$$A(x_1, A(x_2^n, A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2})), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))).$$

Hence, since (Q,A) is an $n-$group, we conclude that for every $x_{n+1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}), x_{n+1}) = A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2})),$$

hence, by Th. 1.7 from Chapter X, we conclude that $\mathbf{e}_B$ is a central operation of the $n-$group (Q,A). (For $n=2$ $x_{n+1}^{2n-2} = \emptyset$.)

Sketch of the proof of 3° :

Puting $x_2^{n-1} = a_1^{n-2}, x_1 = \mathbf{e}_B(a_1^{n-2})$ and $x_n = x$ in (1) we obtain

$$x = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))),$$

hence, by Th.1.3 from Chapter III, we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the following implications hold

$$x = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) \Rightarrow$$

$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$

$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})))) \Rightarrow$

$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$

$A(A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) \overset{1.3III}{\Longrightarrow}$

$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$

$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) \overset{1.3II}{\Longrightarrow}$

$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) = A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})),$

i.e. the following equality holds

$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) = A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})).$$

Hence, by the substitution $x = \mathbf{e}(a_1^{n-2})$, we conclude that for every sequence a_1^{n-2} over Q the following equality holds

$$(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1} = \mathbf{e}_B(a_1^{n-2}).$$

Finally, by $1° - 3°$we conclude that Theorem 3.1 holds.[1] □

3.2. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $n \geq 3$, $A \in \sum$, $|\sum| \geq 2$, and let for all $B \in \sum$ there be a central operation $\boldsymbol{\alpha}$ of the $n-$group (Q, A) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$ *and*
$(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}))^{-1} = \boldsymbol{\alpha}(a_1^{n-2}),$
where $^{-1}$ *is an inverse operation in the $n-$group (Q, A). Then $(Q, \sum)$ is a SAAnQ of the type XX.*

Proof. The following statements hold:

°1 If $B \in \sum$, then (Q, B) is an $n-$quasigroup; and

°2 For all $i \in \{2, \ldots, n\}$, for every $x_1^{2n-1} \in Q$ and for every $B, C \in \sum$ the following equality holds

[1] For $B = A, \boldsymbol{\alpha} = \mathbf{e}$.

$B(B(x_1^n), x_{n+1}^{2n-1}) = C(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1})$.

The proof of $^\circ 1$:

Let B be an arbitrary operation from $\sum$. Also, let $\boldsymbol{\alpha}$ be a central operation of the $n-$group (Q, A) such that for all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$.

Hence, by Prop. 1.9 from Chapter X, by Def. 1.10 from Chapter X, and Def. 1.1 from Chapter I, we conclude that the statement $^\circ 1$ holds.

The proof of $^\circ 2$:

Let B and C be arbitrary operations from $\sum$. Also, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the $n-$group (Q, A) such that for all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold:

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$ and

$C(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_n))$.

Further on, let permutation α $[\beta]$ be associated to the central operation $\boldsymbol{\alpha}$ $[\boldsymbol{\beta}]$ of the $n-$group (Q, A). Hence, by Th. 1.11 from Chapter X and by Th. 1.13 from Chapter X, we conclude that for every $x_1^n \in Q$ the following equalities hold:

$$\begin{aligned}
B(B(x_1^n), x_{n+1}^{2n-1}) &\overset{1.11X}{=} A(A(x_1^{n-1}, \alpha(x_n)), x_{n+1}^{2n-2}, \alpha(x_{2n-1}))\\
&\overset{1.11X}{=} A(\alpha A(x_1^{n-1}, x_n), x_{n+1}^{2n-2}, \alpha(x_{2n-1}))\\
&\overset{1.11X}{=} \alpha(\alpha A(A(x_1^n), x_{n+1}^{2n-1}))\\
&\overset{1.13X}{=} A(A(x_1^n), x_{n+1}^{2n-1}), \text{ and}\\
C(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1}) &\overset{1.11X}{=} A(\beta(x_1), x_2^{i-1}, A(\beta(x_i), x_{i+1}^{i+n-1}), x_{i+n}^{2n-1})\\
&\overset{1.11X}{=} \beta(\beta A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}))\\
&\overset{1.13X}{=} A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}).
\end{aligned}$$

Since (Q, A) is an $n-$group, we conclude that the statement $^\circ 2$ holds.

Finally, by $^\circ 1, ^\circ 2$ and by Def. 2.2, we conclude that the theorem holds. □

3.3 Remark: *$SAA2Q$ of the type XX was described in [Belousov 1965].*

Cf. Th. 3.1 with Th. 1.3 $[-(v)]$.

4 A description of an $SAAnQ$ of the type XY with condition $(\forall i)X_1 = X_{2i-1}$

4.1. Theorem *[Ušan 2001/1]: Let* $(Q, \sum)$ *be an* $SAAnQ$ *of the type* XY *with condition* $(\forall i)X_1 = X_{2i-1}$ *and* $n \geq 3$*. Also, let* A *be an arbitrary operation from* $\sum$ *[cf. Prop. 1.2]. Then, for all* $B \in \sum$ *there is a central operation* $\boldsymbol{\alpha}$ *of the* $n-$*group* (Q, A) *such that for every* $x_1^n \in Q$ *and for every sequence* a_1^{n-2} *over* Q *the following equality holds:*

$$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Proof. Let A and B be two arbitrary operations from $\sum$. By Proposition 1.2, (Q, A) and (Q, B) are $n-$groups. By Th. 2.6 from Chapter II, (Q, A) and (Q, B) have $\{1, n\}-$neutral operations, denoted, respectively, by $\mathbf{e}$ and $\mathbf{e}_B$. Let also the inverse operation in (Q, A) be denoted $^{-1}$ (cf. Chapter III).

The following statements hold:

1° For all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q we have

(1) $\quad B(x_1^n) = A(x_1^{n-1}, A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x_n))$; and

2° For all $x \in Q$ and for every sequences a_1^{n-2} and b_1^{n-2} over Q the following holds

$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) = A(x, (b_1^{n-2}, \mathbf{e}_B(b_1^{n-2}))^{-1}, b_1^{n-2}),$$

i.e. $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}(c_1^{n-2}) \overset{def}{=} (c_1^{n-2}, \mathbf{e}_B(c_1^{n-2}))^{-1}$, is a central operation of the $n-$group (Q, A).

The proof of 1°.

By Def. 2.2 and by condition $(\forall i)X_1 = X_{2i-1}$, for every $x_1^{2n-1} \in Q$ the following equality holds

$A(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{n-1}, B(x_n^{2n-1}))$,

hence, by the substitutions $x_{n+1}^{2n-2} = a_1^{n-2}$ and $x_{2n-1} = \mathbf{e}(a_1^{n-2})$, where a_1^{n-2}

is an arbitrary sequence over Q, we conclude that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

(a) $\quad B(x_1^n) = A(x_1^{n-1}, B(x_n, a_1^{n-2}, \mathbf{e}(a_1^{n-2})))$.

Putting $x_2^{n-1} = a_1^{n-2}$, $x_1 = \mathbf{e}_B(a_1^{n-2})$ and $x_n = x$ in (a), we obtain

$x = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})))$,

hence, by Th. 1.3 from Chapter III, we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the following implications hold

$$x = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))) \Rightarrow$$
$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$$
$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})))) \Rightarrow$$
$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$$
$$A(A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), a_1^{n-2}, B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))) \Rightarrow$$
$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$$
$$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))) \Rightarrow$$
$$A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x) =$$
$$B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})),$$

i.e. the following equality holds

(b) $B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1}, a_1^{n-2}, x)$.

Putting (b) in (a) we obtain 1°.

The proof of 2°.

Let

(c) $\boldsymbol{\alpha}(c_1^{n-2}) \overset{def}{=} (c_1^{n-2}, \mathbf{e}_B(c_1^{n-2}))^{-1}$

for every sequence c_1^{n-2} over Q. By the substitutions (c), the formula (1) reduced to

(1') $B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$.

By Def. 2.2 and by condition $(\forall i) X_1 = X_{2i-1}$, for every $x_1^{2n-1} \in Q$ the following equality holds

$$A(x_1^{n-2}, B(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, B(x_n^{2n-1})).$$

Hence, by $1°[(1')]$, we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_{2n-2})), x_{2n-1}) =$$
$$A(x_1^{n-1}, A(x_n^{2n-2}, A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x_{2n-1}))) \Rightarrow$$
$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_{2n-2})), x_{2n-1}) =$$
$$A(x_1^{n-1}, A(x_n^{2n-3}, A(x_{2n-2}, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}), x_{2n-1})) \Rightarrow$$
$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_{2n-2})), x_{2n-1}) =$$
$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(x_{2n-2}, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})), x_{2n-1}) \Rightarrow$$
$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_{2n-2}) = A(x_{2n-2}, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}),$$

hence, by Def. 1.1 from Chapter X, we conclude that $\boldsymbol{\alpha}$ *[*cf. (c)*]* is a central operation of the $n-$group (Q, A).

Finally, by 1° and 2°, we conclude that Theorem 4.1 holds.[2] □

4.2. Theorem *[Ušan 2001/1]: Let (Q, A) be an $n-$group, $n \geq 3$, $A \in \sum, |\sum| \geq 2$, and let for all $B \in \sum$ there exist a central operation $\boldsymbol{\alpha}$ of the $n-$group (Q, A) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds*

$$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Then $(Q, \sum)$ is an $SAAnQ$ of the type XY with condition $(\forall i \in \{2, \ldots, n\}) X_1 = X_{2i-1}$.

Proof. The following statements hold:

°1 If $B \in \sum$, then (Q, B) is an $n-$quasigroup; and

°2 For all $i \in \{2, \ldots, n\}$, for every $x_1^{2n-1} \in Q$ and for every $B, C \in \sum$ the following equality holds

[2]For $B = A$, $\boldsymbol{\alpha} = \mathbf{e}$

$B(C(x_1^n), x_{n+1}^{2n-1}) = B(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1})$.

The proof of °1 :

Let B be an arbitrary operation from $\sum$. Also, let $\boldsymbol{\alpha}$ be a central operation of the $n-$group (Q, A) such that for all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$.
Hence, by Prop. 1.9 from Chapter X, by Def. 1.10 from Chapter X, and Def. 1.1 from Chapter I, we conclude that the statement °1 holds.

The proof of °2 :

Let B and C be arbitrary operations from $\sum$. Also, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the $n-$group (Q, A) such that for all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$ and

$C(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_n))$.
Further on, let permutation α $[\beta]$ be associated to the central operation $\boldsymbol{\alpha}$ $[\boldsymbol{\beta}]$ of the $n-$group (Q, A). Hence, by Th. 1.11 from Chapter X, we conclude that for every $x_1^{2n-1} \in Q$ the following equalities hold:

$$\begin{aligned}
B(C(x_1^n), x_{n+1}^{2n-1}) &\overset{1.11X}{=} A(A(x_1^{n-1}, \beta(x_n)), x_{n+1}^{2n-2}, \alpha(x_{2n-1}))\\
&\overset{1.11X}{=} \alpha(A(\beta A(x_1^n), x_{n+1}^{2n-1}))\\
&\overset{1.11X}{=} \alpha(\beta A(A(x_1^n), x_{n+1}^{2n-1})) \quad \text{and}\\
B(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1}) &\overset{1.11X}{=} A(\alpha(x_1), x_2^{i-1}, A(x_i^{i+n-2}, \beta(x_{i+n-1})), x_{i+n}^{2n-1})\\
&\overset{1.11X}{=} \alpha(A(x_1^{i-1}, \beta A(x_i^{i+n-1}), x_{i+n}^{2n-1}))\\
&\overset{1.11X}{=} \alpha(\beta A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1})).
\end{aligned}$$

Since (Q, A) is an $n-$group, we conclude that the statement °2 holds.

Finally, by °1, °2 and Def. 2.2, we conclude that Th. 4.2 holds. □

4.3. Remark: *SAA2Q of the type XY with condition* $(\forall i)X_1 = X_{2i-1}$ *was described in [Belousov 1965]. Cf. Th. 4.1 with Th. 1.3 [−(iv)].*

5 A description of an $SAAnQ$ of the type XY with condition $(\exists i \in \{2,\ldots,n\})X_{2(i-1)-1} \neq X_{2i-1}$

5.1. Theorem *[Ušan 2001/1]: Let $(Q,\sum)$ be an $SAAnQ$ of the type XY with condition $(\exists i \in \{2,\ldots,n\})X_{2(i-1)-1} \neq X_{2i-1}$ and $n \geq 3$. Also, let C be an arbitrary operation from $\sum$ [cf. Prop. 1.2]. Then, for all $D \in \sum$ there is a central operation $\boldsymbol{\alpha}$ of the $n-$group (Q,C) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds:*

$$D(x_1^n) = C(x_1^{n-1}, C(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Proof. Let A and B be two arbitrary operations from $\sum$. By Prop. 1.2, (Q,A) and (Q,B) are $n-$groups. By Th. 2.6 from Chapter II, (Q,A) and (Q,B) have $\{1,n\}-$neutral operations, denoted, respectively, by $\mathbf{e}_A$ and $\mathbf{e}_B$. Let also the inverse operation in (Q,A) be denoted by ^{1_A}, and the inverse operation in (Q,B) be denoted by ^{1_B}, (cf. Chapter III). By Def. 2.2 and by condition $(\exists i \in \{2,\ldots,n\})X_{2(i-1)-1} \neq X_{2i-1}$, we conclude that there is an $j \in \{1,\ldots,n-1\}$ such that for every $x_1^{2n-1} \in Q$ the following equality holds
(1) $A(x_1^{j-1}, B(x_j^{j+n-1}), x_{j+n}^{2n-1}) = B(x_1^j, A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1})$.

Firstly we observe that under the assumption the following statements hold:

1° For all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

(1_A) $A(x_1^n) = B(x_1^{j-1}, B(x_j, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B}), x_{j+1}^n)$ and

(1_B) $B(x_1^n) = A(x_1^j, A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1_A}), a_1^{n-2}, x_{j+1}), x_{j+2}^n)$; and

2° For all $x \in Q$ and for every sequences a_1^{n-2} and b_1^{n-2} over Q the following equalities hold

$(\widehat{1}_A)$ $B(x, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B}) = B(b_1^{n-2}, (b_1^{n-2}, \mathbf{e}_A(b_1^{n-2}))^{-1_B}, x)$,

$(\widehat{1}_B)$ $A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1_A}, a_1^{n-2}, x) = A(x, (b_1^{n-2}, \mathbf{e}_B(b_1^{n-2}))^{-1_A}, b_1^{n-2})$,
i.e. $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}(c_1^{n-2}) \overset{def}{=} (c_1^{n-2}, \mathbf{e}_A(c_1^{n-2}))^{-1_B}$, is a central operation of the $n-$group (Q, B) and $\boldsymbol{\beta}$, where $\boldsymbol{\beta}(c_1^{n-2}) \overset{def}{=} (c_1^{n-2}, \mathbf{e}_B(c_1^{n-2}))^{-1_A}$, is a central operation of the $n-$group (Q, A).

The proof of 1° :

By the substitutions $x_{j+1}^{j+n-2} = a_1^{n-2}$ and $x_{j+n-1} = \mathbf{e}(a_1^{n-2})$ in (1), where a_1^{n-2} is an arbitrary sequence over Q, we conclude that for every $x_1^j, a_1^{n-2}, x_{j+n}^{2n-1} \in Q$ the following equality holds
(a) $A(x_1^{j-1}, B(x_j, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})), x_{j+n}^{2n-1}) = B(x_1^j, x_{j+n}^{2n-1})$.

By Th. 1.3 from Chapter III and by Def. 1.1 from Chapter I, we conclude that for every $a_1^{n-2}, u, x_j \in Q$ the following equivalence holds
(b) $B(x_j, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})) = u \Leftrightarrow x_j = B(u, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B})$.

By (a) and (b), we conclude that for every $y_1^n, a_1^{n-2} \in Q$ the following equality holds
$A(y_1^n) = B(y_1^{j-1}, B(y_j, a_1^{n-2}, (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B}), y_{j+1}^n)$,
i.e. the equality (1_A) holds.

Similarly, if we put in (1) $x_{j+2}^{j+n-1} = a_1^{n-2}$ and $x_{j+1} = \mathbf{e}_B(a_1^{n-2})$, we conclude that for every $y_1^n, a_1^{n-2} \in Q$ the equality
$B(y_1^n) = A(y_1^j, A((a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1_A}, a_1^{n-2}, y_{j+1}), y_{j+2}^n)$
holds.

The proof of 2° :

We distinguish the following cases:

Case I:

(I_1) $\qquad A(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$ and

(I_2) $\qquad B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}) = B(x_1^2, A(x_3^{n+2}), x_{n+3}^{2n-1})$.

Case II:

(II_1) $\qquad A(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1})$ and

(II_2) $\quad A(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1}) = B(x_1^2, A(x_3^{n+2}), x_{n+3}^{2n-1})$.

Case III:

(III_1) $\quad A(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$ and

(III_2) $\quad B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}) = A(x_1^2, B(x_3^{n+2}), x_{n+3}^{2n-1})$.

Case IV:

(IV_1) $\quad A(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1})$,

(IV_2) $\quad A(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1}) = A(x_1^2, B(x_3^{n+2}), x_{n+3}^{2n-1})$ and

(IV_3) $\quad$ (1) for some $j \in \{3, \ldots, n-1\}$.

The proof of 2°–I:

In case I the formulas (1) and (1_B), respectively, reduced to
(I_1) *[*(1) for $j = 1$*]* and
$(1'_A)$ $A(x_1^n) = B(B(x_1, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x_2^n)$ *[*(1_A) for $j = 1$*]*,
where $\boldsymbol{\alpha}(a_1^{n-2}) \overset{def}{=} (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B}$.

By $(1'_A)$ and by (I_2), we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

$$B(x_1, B(B(x_2, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x_3^{n+1}), x_{n+2}^{2n-1}) =$$
$$B(x_1^2, B(B(x_3, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_4^{n+2}), x_{n+3}^{2n-1}) \Rightarrow$$
$$B(x_1, B(x_2, B(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x_3), x_4^{n+1}), x_{n+2}^{2n-1}) =$$
$$B(x_1^2, B(B(x_3, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_4^{n+2}), x_{n+3}^{2n-1}) \Rightarrow$$
$$B(x_1^2, B(B(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x_3), x_4^{n+2}), x_{n+3}^{2n-1}) =$$
$$B(x_1^2, B(B(x_3, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_4^{n+2}), x_{n+3}^{2n-1}) \Rightarrow$$
$$B(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x_3) = B(x_3, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})),$$

hence, by Th. 1.7 from Chapter X, we conclude that $\boldsymbol{\alpha}$ is a central operation of the $n-$group (Q, B).

The proof of 2°–II:

In case II the formulas (1) and (1_A), respectively, reduced to (II_2) /(1) for $j = 2$/ and
$(1'_B)$ $B(x_1^n) = A(x_1^2, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_3), x_4^n)$ /(1_B) for $j = 2$/,
where $\boldsymbol{\beta}(a_1^{n-2}) \overset{def}{=} (a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))^{-1_A}$.

By $(1'_B)$ and by (II_1), we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

$$A(A(x_1^2, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_3), x_4^n), x_{n+1}^{2n-1}) =$$

$$A(x_1, A(x_2^3, A(\boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}, x_4), x_5^{n+1}), x_{n+2}^{2n-1}) \Rightarrow$$

$$A(A(x_1^2, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_3), x_4^n), x_{n+1}^{2n-1}) =$$

$$A(x_1, A(x_2, A(x_3, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}), x_4^{n+1}), x_{n+2}^{2n-1}) \Rightarrow$$

$$A(A(x_1^2, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_3), x_4^n), x_{n+1}^{2n-1}) =$$

$$A(A(x_1^2, A(x_3, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}), x_4^n), x_{n+1}^{2n-1}) \Rightarrow$$

$$A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_3) = A(x_3, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}),$$

hence, by Def. 1.1 from Chapter X, we conclude that $\boldsymbol{\beta}$ is a central operation of the $n-$group (Q, A).

The proof of 2°–III:

In case III the formulas (1) and (1_A), respectively, reduced to
(III_1) /(1) for $j = 1$/ and
$(1'_A)$ $A(x_1^n) = B(B(x_1, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x_2^n)$ /(1_A) for $j = 2$/,
where $\boldsymbol{\alpha}(a_1^{n-2}) \overset{def}{=} (a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))^{-1_B}$.

By $(1'_A)$ and by (III_2), we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following implications hold

$$B(x_1, B(B(x_2, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x_3^{n+1}), x_{n+2}^{2n-1}) =$$

$$B(B(x_1, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_2, B(x_3^{n+2}), x_{n+3}^{2n-1}) \Rightarrow$$

$B(x_1, B(x_2, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), B(x_3^{n+2}), x_{n+3}^{2n-1}) =$

$B(x_1, B(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x_2), B(x_3^{n+2}), x_{n+3}^{2n-1}) \Rightarrow$

$B(x_2, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = B(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x_2),$

hence, by Th. 1.7 from Chapter X, we conclude that $\boldsymbol{\alpha}$ is a central operation of the $n-$group (Q, B).

The proof of 2°–IV:

In case *IV* the formulas (1) and (1_B), reduced to (1) and (1_B) for some $j \in \{3, \dots, n\}$. Furthermore, let
$\boldsymbol{\beta}(c_1^{n-2}) \overset{def}{=} (c_1^{n-2}, \mathbf{e}_B(c_1^{n-2}))^{-1_A}$.

By (1_B) and by (IV_1), we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

$A(A(x_1^j, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_{j+1}), x_{j+2}^n), x_{n+1}^{2n-1}) =$

$A(x_1, A(x_2^{j+1}, A(\boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}, x_{j+2}), x_{j+3}^{n+1}), x_{n+2}^{2n-1}) \Rightarrow$

$A(A(x_1^j, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_{j+1}), x_{j+2}^n), x_{n+1}^{2n-1}) =$

$A(x_1, A(x_2^j, A(x_{j+1}, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}), x_{j+2}^{n+1}), x_{n+2}^{2n-1}) \Rightarrow$

$A(A(x_1^j, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_{j+1}), x_{j+2}^n), x_{n+1}^{2n-1}) =$

$A(A(x_1^j, A(x_{j+1}, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}), x_{j+2}^n), x_{n+1}^{2n-1}) \Rightarrow$

$A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_{j+1}) = A(x_{j+1}, \boldsymbol{\beta}(b_1^{n-2}), b_1^{n-2}),$

hence, by Def. 1.1 from Chapter X, we conclude that $\boldsymbol{\beta}$ is a central operation of the $n-$group (Q, A).

The proof 2° is completed.

Finally, by $1°, 2°$ and by Prop. 1.3 from Chapter X, we conclude that the proposition is satisfied. □

5.2. Theorem *[Ušan 2001/1]: Let* $(Q, \sum)$ *be an* $n-$*group,* $n \geq 3$, $A \in \sum$, $|\sum| \geq 2$ *and let for all* $B \in \sum$ *there exist a central operation* $\boldsymbol{\alpha}$ *of the*

$n-$group (Q, A) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Then $(Q, \sum)$ is an $SAAnQ$ of the type XY with condition $(\exists i \in \{2, \ldots, n\})$ $X_{2(i-1)-1} \neq X_{2i-1}$.

Proof. The following statements hold:

$^\circ1$ If $B \in \sum$, then (Q, B) is an $n-$quasigroup;

$^\circ2$ For all $i \in \{2, \ldots, n\}$, for every $x_1^{2n-1} \in Q$ and for every $B, C \in \sum$ the following equality holds

$$B(C(x_1^n), x_{n+1}^{2n-1}) = B(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1});$$ and

$^\circ3$ For all $j \in \{0, \ldots, n-1\}$, for every $x_1^{2n-1} \in Q$ and for every $B, C \in \sum$ the following equality holds

$$B(x_1^j, C(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}) = C(x_1^j, B(x_{j+1}^{i+n}), x_{i+n+1}^{2n-1}).$$

The proof of $^\circ1$: The proof of $^\circ1$ from Th. 4.2.

The proof of $^\circ2$: The proof of $^\circ2$ from Th. 4.2.

The proof of $^\circ3$:

Let B and C be arbitrary operations from $\sum$. Also, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the $n-$group (Q, A) such that for all $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold:

$$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n))$$ and

$$C(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Further on, let permutation α $[\beta]$ be associated to the central operation $\boldsymbol{\alpha}$ $[\boldsymbol{\beta}]$ of the $n-$group (Q, A). Hence, by Th. 1.11 from Chapter X and by Th. 1.12 from Chapter X, we conclude that for every $x_1^{2n-1} \in Q$ the following equalities hold:

$$\begin{aligned} B(x_1^j, C(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}) &\overset{1.11X}{=} \alpha A(x_1^j, \beta A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}) \\ &\overset{1.11X}{=} \alpha(\beta A(x_1^j, A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1})) \\ &\overset{1.12X}{=} \beta(\alpha A(x_1^j, A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1})) \\ &\overset{1.11X}{=} \beta A(x_1^j, \alpha A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}) \\ &= C(x_1^j, B(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}). \end{aligned}$$

The proof of $°3$ is complete.

Finally, by $°1 -° 3$ and Def. 2.2, we conclude that Th. 5.2 holds. □

5.3. Remark: *$SAA2Q$ of the type XY with condition $X_1 \neq X_3$ was described in [Belousov 1965]. Cf. Th. 5.1 with Th. 1.3 [−(vi)].*

6 A description of an $SAAnQ$ in terms of Hosszú-Gluskin algebras

A consequence of Theorem 3.1, of Theorem 2.2 from Chapter X and of Theorem 2.3 from Chapter X is the following proposition:

6.1. Theorem *[Ušan 2001/1]: Let $(Q, \sum)$ be an $SAAnQ$ of the type XX and $n \geq 3$. Also, let A be an arbitrary operation from $\sum$ and $(Q, \{\cdot, \varphi, b\})$ an $nHG-$algebra associated to the $n-$group (Q, A). Then, for every $B \in \sum$ there is exactly one $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold:*

(a) $B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n,$

(b) $(a \cdot b) \cdot x = x \cdot (a \cdot b),$

(c) $\varphi(a) = a$ *and*

(d) $(a \cdot b) \cdot (a \cdot b) = e,$

where e is the neutral element of the group $(Q, \cdot)$.

A consequence of Theorem 4.1, Theorem 5.1 and Theorem 2.2 from Chapter X is the following proposition:

6.2. Theorem *[Ušan 2001/1]: Let $(Q,\sum)$ be an $SAAnQ$ of the type XY and $n \geq 3$. Also, let A be an arbitrary operation from $\sum$ and $(Q,\{\cdot,\varphi,b\})$ an $nHG-$algebra associated to the $n-$group (Q,A). Then, for every $B \in \sum$ there is exactly one $a \in Q$ such that for every $x, x_1^n \in Q$ the equalities (a)–(c) from Th. 6.1 hold.*

6.3. Theorem *[Ušan 2001/1]: Let $n \geq 3$, let (Q,A) be an $n-$group, $(Q,\{\cdot,\varphi,b\})$ its associated $nHG-$algebra, $A \in \sum$, $|\sum| \geq 2$, and let for all $B \in \sum$ there is an element $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold*

$$\varphi(a) = a,$$
$$(a \cdot b) \cdot x = x \cdot (a \cdot b) \text{ and}$$
$$B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n.$$

Then $(Q,\sum)$ is a $SAAnQ$ of the type XY.

Proof. By Th. 4.2, Th. 5.2 and by Th. 2.4 from Chapter X. □

6.4. Theorem *[Ušan 2001/1]: Let $n \geq 3$, let (Q,A) be an $n-$group, $(Q,\{\cdot,\varphi,b\})$ its associated $nHG-$algebra, $A \in \sum$, $|\sum| \geq 2$, and let for all $B \in \sum$ there is an element $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold*

$$\varphi(a) = a,$$
$$(a \cdot b) \cdot x = x \cdot (a \cdot b),$$
$$(a \cdot b) \cdot (a \cdot b) = e \text{ [e the neutral element of the group $(Q,\cdot)$] and}$$
$$B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_{n-1} \cdot b \cdot a \cdot b \cdot x_n.$$

Then $(Q,\sum)$ is a $SAAnQ$ of the type XX.

Proof. By Th. 3.2 and by Th. 2.5 from Chapter X. □

7 On a description of the case $n = 3$ by Yu. M. Movsisyan

$SAA3Q$ were described firstly by Yu. M. Movsisyan in 1984 (cf. [Movsisyan 1986]).

In this section we compare one proposition of Yu. M. Movsisyan /[Movsisyan 1986] p. 152, direction " $\Rightarrow$ " of Th. 2.2.37/ with the corresponding proposition from 6 for $n = 3$ /Th. 6.2 for $n = 3$/. Therefore, we advance the following definition:

7.1. Definition *[Ušan 2001/1]: Let $(Q, \{\cdot, \beta, r, s, t\})$ be an algebra, where $\cdot$ is a binary operation in Q, β is a permutation of the set Q, and r, s, t fixed elements of the set Q. Then we say that $(Q, \{\cdot, \beta, r, s, t\})$ is a $3M-$algebra iff the following statements hold:*
(1) $(Q, \cdot)$ *is a group;*
(2) $\beta \in Aut(Q, \cdot)$*;*
(3) $\beta(s \cdot r) = r \cdot s \cdot t^{-1}$*, where* $^{-1}$ *is the inverse operation in the group* $(Q, \cdot)$*; and*
(4) $(\forall x \in Q)\beta^2(x) \cdot (\beta(r^{-1}) \cdot s) = (\beta(r^{-1}) \cdot s) \cdot x$.

Using Def. 7.1, Theorem of Movsisyan corresponding Th. 6.2 for $n = 3$ /[Movsisyan 1986], p. 152, direction " $\Rightarrow$ " of Theorem 2.2.37/, can be formulated in the following way:

7.2. Theorem *[Movsisyan 1984]: Let $(Q, \sum)$ be an $SAA3Q$ of the type XY. Then there exists a $3M-$algebra $(Q, \{\cdot, \beta, r, s, t\})$ such that for every $B \in Q$ there is exactly one $p \in Q$ such that for every $x, x_1^3 \in Q$ the following equalities are satisfied*
$(\overline{a})$ $B(x_1^3) = x_1 \cdot r \cdot \beta(x_2) \cdot s \cdot p \cdot x_3,$
$(\overline{b})$ $p \cdot x = x \cdot p$ *and*
$(\overline{c})$ $\beta(p) = t \cdot p.$

Theorem 6.2 can be formulated in a similar way for $n = 3$:

7.3. Proposition *Let $(Q, \sum)$ be an $SAA3Q$ of the type XY. Then there exists a $3HG-$algebra $\{Q, \{\cdot, \varphi, b\})$ such that for every $B \in \sum$ there is exactly one $a \in Q$ such that for every $x, x_1^3 \in Q$ the following equalities are satisfied*

$(\widehat{a})$ $B(x_1^3) = x_1 \cdot \varphi(x_2) \cdot b \cdot a \cdot b \cdot x_3,$

$(\widehat{b})$ $(a \cdot b) \cdot x = x \cdot (a \cdot b)$ *and*

$(\widehat{c})$ $\varphi(a) = a.$

8 On congruences in a SAAnQ

8.1. Theorem *[Ušan, Žižović 2004]: Let $(Q; \sum)$ be a super-associative algebra with $n-$quasigroup operations $(n \geq 3)$ [cf. Def. 1.1] and let A be an arbitrary element of $\sum$. Then, the following equality holds*

$$Con(Q; \textstyle\sum) = Con(Q; A).$$

Proof. Let $(Q; \sum)$ be a super-associative algebra with $n-$quasigroup operations and let A be an arbitrary element of $\sum$. Also, let $(Q; \cdot, \varphi, b)$ be an arbitrary $nHG-$algebra associated to the $n-$group $(Q; A)$ *[*cf. Prop. 1.2-XI and Def. 2.3-IV*]*. Then, by Th. 6.1-XI and Th. 6.2-XI, for every $B \in \sum$ there is exactly one $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold

$$B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n,$$
$$(a \cdot b)x = x \cdot (a \cdot b) \text{ and}$$
$$\varphi(a) = a.$$

Whence, we have: $(Q; \cdot, \varphi, b \cdot a \cdot b)$ is an $nHG-$algebra associated to the $n-$group $(Q; B)$.

Finally, by Th. 3.1-VI, we conclude that the proposition is satisfied. □

Chapter XII

NOTE ON $(k(n-1)+1)-$SEMIGROUPS

1 Auxiliary propositions

1.1. Proposition *[Dudek 1995]: Let $n \geq 3$, let (Q, A) be an $n-$groupoid and let E be an $(n-2)-$ary operation in Q. Also, let for all $x, x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold:*

(i) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

(ii) $A(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2})) = x$ *and*

(iii) $A(b_1^{n-2}, \mathsf{E}(b_1^{n-2}), x) = x$.

Then (Q, A) is an $n-$group.

Proof.[1] Firstly we observe that under the assumption the following statements hold:

1° For every $x, y, a, a_1^{n-2} \in Q$ the implication holds

$$A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow x = y;$$

2° (Q, A) is an $n-$semigroup;

3° $(\forall a \in Q)(\forall c_i \in Q)_1^{n-3} a = \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))$;

4° For every $x, y, a, a_1^{n-2} \in Q$ the implication

$$A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \Rightarrow x = y$$

holds;

5° For every $x, y, a, a_1^{n-2} \in Q$ the equivalence

$$A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a) \Leftrightarrow x = y$$

holds; and

[1] [Ušan, Žižović 2002/1].

6° For every $x, a, b, a_1^{n-2} \in Q$ and for all sequence c_1^{n-3} over Q

$$A(a, x, a_1^{n-2}) = b \Leftrightarrow$$
$$x = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})).$$

Sketch of the proof of 1° :

$$A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow$$
$$A(A(x, a, a_1^{n-2}), \mathsf{E}(a_1^{n-2}), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) =$$
$$A(A(y, a, a_1^{n-2}), \mathsf{E}(a_1^{n-2}), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) \overset{(i)}{\Rightarrow}$$
$$A(x, A(a, a_1^{n-2}, \mathsf{E}(a_1^{n-2})), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) =$$
$$A(y, A(a, a_1^{n-2}, \mathsf{E}(a_1^{n-2})), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) \overset{(ii)}{\Rightarrow}$$
$$A(x, a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) = A(y, a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) \overset{(ii)}{\Rightarrow}$$
$$x = y.$$

The proof of 2° :

By 1° and by Prop. 2.1 from Chapter III.

Sketch of the proof of 3° :

$$A(a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))) \overset{(iii)}{=} \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})),$$
$$A(a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))) \overset{(ii)}{=} a.$$

Sketch of the proof of 4° :

$$A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \Rightarrow$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(a, x, a_1^{n-2}), \mathsf{E}(a_1^{n-2})) =$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(a, y, a_1^{n-2}), \mathsf{E}(a_1^{n-2})) \overset{2^\circ}{\Rightarrow}$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, A(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2}))) =$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, A(y, a_1^{n-2}, \mathsf{E}(a_1^{n-2}))) \overset{(ii)}{\Rightarrow}$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, x) = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, y) \overset{3^\circ}{\Rightarrow}$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), x) =$$
$$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), y) \overset{(iii)}{\Rightarrow} x = y.$$

Sketch of the proof of 5° :

$$A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a) \Rightarrow$$

$A(d_1^2, A(a_1^{n-2}, x, a), d_3^{n-1}) = A(d_1^2, A(a_1^{n-2}, y, a), d_3^{n-1}) \overset{2^\circ}{\Rightarrow}$
$A(A(d_1^2, a_1^{n-2}), x, a, d_3^{n-1}) = A(A(d_1^2, a_1^{n-2}), y, a, d_3^{n-1}) \overset{4^\circ}{\Rightarrow}$
$x = y.$

Sketch of the proof of 6° :
$A(a, x, a_1^{n-2}) = b \overset{5^\circ}{\Leftrightarrow}$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(a, x, a_1^{n-2}), \mathsf{E}(a_1^{n-2})) =$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})) \overset{2^\circ}{\Leftrightarrow}$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, A(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2}))) =$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})) \overset{(ii)}{\Leftrightarrow}$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), a, x) =$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})) \overset{3^\circ}{\Leftrightarrow}$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), x) =$
$A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})) \overset{(iii)}{\Longleftrightarrow}$
$x = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})).$

Finally, considering $2^\circ, 4^\circ$ and 6°, by Th.3.4 from Chapter IX, we conclude that (Q, A) is an $n-$group. □

Similarly, one could prove also the following proposition:

1.2. Proposition *[Dudek 1995]: Let $n \geq 3$, let (Q, A) be an $< n-1, n >$ $-$associative $n-$groupoid and let E be an $(n-2)-$ary operation in Q. In addition, let for all $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold*

$$A(\mathsf{E}(a_1^{n-2}), a_1^{n-2}, x) = x \text{ and } A(x, \mathsf{E}(b_1^{n-2}), b_1^{n-2}) = x.$$

Then (Q, A) is an $n-$group.

1.3.Remark: *E from 1.1 and from 1.2 is an $\{1, n\}-$neutral operation of the $n-$group (Q, A). [Cf. Th. 2.6 from Chapter II and Def. 1.1 from Chapter I.]*

2 $(n-2)-$neutral operations of $(k(n-1)+1)-$groupoids

2.1. Definition *[Ušan 1998/3]: Let $(k,n) \in N \times (N\backslash\{1\})$, let A be a $(k(n-1)+1)-$ary operation in Q and E a mapping of the set Q^{n-2} into the set Q. Then: 1) we say that E is a* **left** *$(n-2)-$* **neutral operation** *of a $(k(n-1)+1)-$***groupoid** *(Q,A) iff for every $\overset{(1)}{a}_1, \ldots, \overset{(1)}{a}_{n-2}, \ldots, \overset{(k)}{a}_1, \ldots, \overset{(k)}{a}_{n-2} \in Q$ the formula*

$$\text{(1)} \qquad \bigwedge_{i=1}^{k} A(\overline{\mathsf{E}(\overset{(j)}{a}{}_1^{n-2}), \overset{(j)}{a}{}_1^{n-2}}\Big|_{j=1}^{i} x, \overline{\mathsf{E}(\overset{(j)}{a}{}_1^{n-2}), \overset{(j)}{a}{}_1^{n-2}}\Big|_{j=i+1}^{k}) = x$$ [2]

holds; 2) we say that E is a **right** *$(n-2)-$***neutral operation** *of a $(k(n-1)+1)-$***groupoid** *(Q,A) iff for every $\overset{(1)}{a}_1, \ldots, \overset{(a)}{a}_{n-2}, \ldots, \overset{(k)}{a}_1, \ldots, \overset{(k)}{a}_{n-2} \in Q$ the formula*

$$\text{(2)} \qquad \bigwedge_{i=1}^{k} A(\overline{\overset{(j)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{a}{}_1^{n-2})}\Big|_{j=1}^{i-1}, x, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{a}{}_1^{n-2})}\Big|_{j=i}^{k}) = x$$

*holds; and 3) we say that E is a $(n-2)-$***neutral operation** *of a $(k(n-1)+1)-$***groupoid** *(Q,A) iff E is a* **left** *$(n-2)-$neutral operation of a $(k(n-1)+1)-$groupoid (Q,A) and a* **right** *$(n-2)-$neutral operation of a $(k(n-1)+1)-$groupoid (Q,A).*

2.2. Remark: *For $n=2$ the formula (1) and the formula (2) reduce, respectively, to the formulas*

$$(\widehat{1}) \qquad \bigwedge_{i=1}^{k} A(\overset{i}{e}, x, \overset{k-i}{e}) = x \ \textit{and}$$

$$(\widehat{2}) \qquad \bigwedge_{i=1}^{k} A(\overset{i-1}{e}, x, \overset{k-i+1}{e}) = x;$$

$e = \mathsf{E}(\emptyset)$. *Further on the conjuction of the formulas $(\widehat{1})$ and $(\widehat{2})$ for all $x \in Q$ is equivalent with the following formula*

$$(e) \qquad \bigwedge_{i=1}^{k+1} A(\overset{i-1}{e}, x, \overset{k-i+1}{e}) = x.$$

---

[2] For example, in [Belousov 1972]: $\{\overset{(i)}{a}{}_p^q\}_{i=t}^{s}$ instead of $\overline{\overset{(i)}{a}{}_p^q}\Big|_{i=t}^{s}$.

(See, also Def. 1.1 and 2 from Chapter II.)

2.3. Proposition *[Ušan 1998/3]: Let* $n \geq 3$ *and let* (Q, B) *be an* $n-$*semigroup with a* $\{1, n\}-$*neutral operation* $\mathbf{e}$.[3] *Further on, let* $k \geq 2$. *Then the following statements hold:*

a) $(Q, \overset{k}{B})$ *is a* $(k(n-1)+1)-$*group; and*

b) $\mathbf{e}$ *is an* $(n-2)-$*operation of the* $(k(n-1)+1)-$*groupoid* $(Q, \overset{k}{B})$.

Proof. 1) By Th. 2.2 from Chapter IX, we conclude that the $n-$semigroup (Q, B) is an $n-$group. Therefore, by 6 from Chapter VI and by Def. 1.1 from Chapter I, we conclude that the statement $a)$ is satisfied.

2) By 1), By Prop. 1.1 from Chapter IV and by 6 from Chapter VI, we conclude that the statement $b)$ holds.

Sketch of a part of the proof of $b)$:

$$\overset{i}{B}(\overline{\overset{(j)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(j)}{a}{}_1^{n-2})}\Big|_{j=1}^{i}, x) =$$

$$\overset{i-1}{B}(\overline{\overset{(j)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(j)}{a}{}_1^{n-2})}\Big|_{j=1}^{i-1}, B(\overset{(i)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(i)}{a}{}_1^{n-2}), x)) =$$

$$\overset{i-1}{B}(\overline{\overset{(j)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(j)}{a}{}_1^{n-2})}\Big|_{j=1}^{i-1}, x). \quad \square$$

[3] See Chapter II and 2.1-IX.

3 Main proposition

3.1. Theorem *[Ušan 1998/3]: Let* $k \geq 2$, $n \geq 2$ *and let* (Q, A) *be a* $(k(n-1)+1)-$*semigroup. Further on, let* E *be a left* $(n-2)-$*neutral operation of a* $(k(n-1)+1)-$*semigroup* (Q, A) **or** *a right* $(n-2)-$*neutral operation of a* $(k(n-1)+1)-$*semigroup* (Q, A)*. Then there exists an* $n-$*groupoid* (Q, B) *such that the following statements hold:*

(*i*) (Q, B) *is an* $n-$*semigroup;*

(*ii*) $A = \overset{k}{B}$;

(*iii*) E *is a* $\{1, n\}-$*neutral operation of the* $n-$*groupoid* (Q, B); *and*

(*iv*) *If* $n \geq 3$*, then* (Q, A) *is a* $(k(n-1)+1)-$*group.*

[Cf. Chapter II-1.]

Proof. 1) Let E be a **right** $(n-2)-$neutral operation of a $(k(n-1)+1)-$semigroup (Q, A); $k \geq 2$, $n \geq 2$. Firstly we observe that under the assumption the following statements hold:

1° Let $\overset{(j)}{a}{}_1^{n-2}$, $j \in \{1, \ldots, n-1\}$, be an arbitrary sequence over Q. Further on, let for every $x_1^n \in Q$

$$(a) \qquad B(x_1^n) \overset{def}{=} A(x_1^n, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{a}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}).$$

Then for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j \in \{1, \ldots, n-1\}$, over Q and for every $x_1^n \in Q$ the following equality holds

$$B(x_1^n) = A(x_1^n, \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}).$$

2° For every $x_1^n \in Q$ and for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j \in \{1, \ldots, n-1\}$, over Q the following equality holds

$$B(x_1^n) = A(x_1^{n-1}, \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}, x_n).$$

3° (Q, B), where the $n-$ary operation B in Q is defined by (a) in 1°, is an $n-$semigroup.

4° For every $x_1^{k(n-1)+1} \in Q$ the following equality holds

$$A(x_1^{k(n-1)+1}) = \overset{k}{B}(x_1^{k(n-1)+1}).$$

5° For all $x \in Q$, for every sequence a_1^{n-2} and for every sequence b_1^{n-2} over Q the following equalities hold

$$B(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2})) = x \text{ and } B(b_1^{n-2}, \mathsf{E}(b_1^{n-2}), x) = x.$$

6° If $n \geq 3$, then (Q, B) is an $n-$group.

7° E is a $\{1, n\}-$neutral operation of the (Q, B).

Sketch of the proof of 1° :

$$B(x_1^n) \overset{2.1}{=} A(B(x_1^n), \overset{(k)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(k)}{a}{}_1^{n-2}), \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}) \overset{(a)}{=}$$

$$A(A(x_1^n, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{a}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}), \overset{(k)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(k)}{a}{}_1^{n-2}), \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}) =$$

$$A(x_1^{n-1}, A(x_n, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{a}{}_1^{n-2})}\,\Big|_{j=1}^{k}), \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}) \overset{2.1}{=}$$

$$A(x_1^n, \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}).$$

Sketch of the proof of 2° :

$$B(x_1^n) \overset{1^\circ}{=} A(x_1^n, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}) \overset{2.1}{=}$$

$$A(x_1^{n-1}, A(\overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}, x_n, c_1^{n-2}, \mathsf{E}(c_1^{n-2})), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}) =$$

$$A(x_1^{n-1}, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2}}\,\Big|_{j=1}^{k-1}, A(x_n, c_1^{n-2}, \mathsf{E}(c_1^{n-2}) \overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1})) \overset{2.1}{=}$$

$$A(x_1^{n-1}, A(\overline{\overset{(j)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}, x_n).$$

The proof of 3° :

Let i be an arbitrary element of the set $\{1,\dots,n-1\}$. Then for every $x_1^{2n-1}\in Q$, for every sequence of sequences $\overset{(j)}{b}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q and for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q the following equality holds

$$A(x_1^{i-1},A(x_i^{i+n-1},\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}),x_{i+n}^{2n-1},\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1})=$$

$$A(x_1^{i},A(x_{i+1}^{i+n-1},\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathsf{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1},x_{i+n}),x_{i+n+1}^{2n-1}\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}),$$

hence, by 1° and 2°, we conclude that for all $i\in\{1,\dots,n-1\}$ and for every $x_1^{2n-1}\in Q$ the following equality holds

$$B(x_1^{i-1},B(x_i^{i+n-1}),x_{i+n}^{2n-1})=B(x_1^{i},B(x_{i+1}^{i+n}),x_{i+n+1}^{2n-1}).$$

Sketch of the proof of 4° :

$A(x_1^{k(n-1)+1})\overset{2.1}{=}$

$A(x_1^{k(n-1)},A(x_{k(n-1)+1},\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1},\overset{(2)}{b}{}_1^{n-2},\mathsf{E}(\overset{(2)}{b}{}_1^{n-2})))=$

$A(x_1^{(k-1)(n-1)},A(x_{(k-1)(n-1)+1}^{k(n-1)+1},\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}),\overset{(2)}{b}{}_1^{n-2}),\mathsf{E}(\overset{(2)}{b}{}_1^{n-2}))\overset{1^\circ,2^\circ}{=}$

$A(x_1^{(k-1)(n-1)},B(x_{(k-1)(n-1)+1}^{k(n-1)+1},\overset{(2)}{b}{}_1^{n-2},\mathsf{E}(\overset{(2)}{b}{}_1^{n-2}))\overset{2.1}{=}$

$A(x_1^{(k-1)(n-1)},A(B(x_{(k-1)(n-1)+1}^{k(n-1)+1}),\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1},\overset{(1)}{b}{}_1^{n-2},\mathsf{E}(\overset{(1)}{b}{}_1^{n-2})),\overset{(2)}{b}{}_1^{n-2},$
$\mathsf{E}(\overset{(2)}{b}{}_1^{n-2}))=$

$A(x_1^{(k-2)(n-1)},A(x_{(k-2)(n-1)+1}^{(k-1)(n-1)},B(x_{(k-1)(n-1)+1}^{k(n-1)+1}),\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}),$
$\overline{\overset{(t)}{b}{}_1^{n-2},\mathsf{E}(\overset{(t)}{b}{}_1^{n-2})}\,\Big|_{t=1}^{2})\overset{1^\circ,2^\circ}{=}$

$A(x_1^{(k-2)(n-1)},B(x_{(k-2)(n-1)+1}^{(k-1)(n-1)},B(x_{(k-1)(n-1)+1}^{k(n-1)+1})),\ \overline{\overset{(t)}{b}{}_1^{n-2},\mathsf{E}(\overset{(t)}{b}{}_1^{n-2})}\,\Big|_{t=1}^{2})\overset{6.3\underline{VI}}{=}$

$A(x_1^{(k-2)(n-1)},\overset{2}{B}(x_{(k-2)(n-1)+1}^{k(n-1)+1}),\ \overline{\overset{(t)}{b}{}_1^{n-2},\mathsf{E}(\overset{(t)}{b}{}_1^{n-2})}\,\Big|_{t=1}^{2}).$

So, for $k=2$ we have:

$$\begin{aligned} A(x_1^{2n-1}) &= A(\overset{2}{B}(x_1^{2n-1}),\ \overline{\overset{(t)}{b}{}_1^{n-2}, \mathsf{E}(\overset{(t)}{b}{}_1^{n-2})}\ \Big|_{t=1}^{2}) \\ &\overset{2.1}{=} \overset{2}{B}(x_1^{2n-1}). \end{aligned}$$

The proof of 5°

By Def. 2.1, we conclude that for all $x \in Q$, for every sequence a^{n-2} over Q and for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j \in \{1, \ldots, k-1\}$, over Q the following equalities hold

$$A(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2}),\ \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\ \Big|_{j=1}^{k-1}) = x \text{ and}$$

$$A(a_1^{n-2}, \mathsf{E}(a_1^{n-2}), x\ \overline{\overset{(j)}{c}{}_1^{n-2}, \mathsf{E}(\overset{(j)}{c}{}_1^{n-2})}\ \Big|_{j=1}^{k-1}) = x,$$

whence, by 1°, we conclude that for all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

$$B(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2})) = x \text{ and } B(a_1^{n-2}, \mathsf{E}(a_1^{n-2}), x) = x.$$

For $n=2$ this equalities reduce to the equalities

$$B(x, \mathsf{E}(\emptyset)) = x \text{ and } B(\mathsf{E}(\emptyset), x) = x.$$

The proof of 6° :

By 3°, by 5° and by Proposition 1.1.

The proof of 7° :

a) For $n=2$: by 5°.

b) For $n \geq 3$: by 6°, by Th. 2.6 from Chapter II, and by Def. 1.1 from Chapter I.

Finally: 1) By 3°, by 4° and by 7°, we conclude that the statements $(i)-(iii)$ hold; and 2) By 4° and by 6°, we conclude that the statement (iv) holds.

Similarly, it is posible to prove also the case: E is a left $(n-2)-$neutral operation of the $(k(n-1)+1)-$semigroup (Q, A). □

Chapter XIII

ON LEFT (RIGHT) DIVISION IN $n-$GROUPS

1 $n-$groups as $n-$groupoids with laws

1.1. Theorem *[Ušan 1997/4]: Let $n \geq 2$ and let (Q, A) be an $n-$group. Furthermore, let $B = ^{-1}A$, where*

(o) $\quad {}^{-1}A(x, z_1^{n-2}, y) = z \overset{def}{\Leftrightarrow} A(z, z_1^{n-2}, y) = x$

for all $x, y, z \in Q$ and for every sequence z_1^{n-2} over Q.[1] *Then the following laws*

(i) $B(B(x, z, b_1^{n-2}), B(y, a_1^{n-2}, z), a_1^{n-2}) = B(x, y, b_1^{n-2})$ *and*

(ii) $B(a, c_1^{n-2}, B(B(B(u, c_1^{n-2}, u), c_1^{n-2}, b), c_1^{n-2}, B(B(v, c_1^{n-2}, v), c_1^{n-2}, a))) = b$

hold in the $n-$groupoid (Q, B).

Proof. Let $n \geq 2$ and let (Q, A) be an $n-$group, $^{-1}$ its inverse operation *[*Chapter III-1*]* and $\mathbf{e}$ its $\{1, n\}-$neutral operation *[*Chapter II-2*]*.

The proof of (i) :

$a)$ By (o), we have

(a) $\quad A(y, a_1^{n-2}, z) = u \Leftrightarrow ^{-1}A(u, a_1^{n-2}, z) = y$ and

(b) $\quad A(x, u, b_1^{n-2}) = v \Leftrightarrow ^{-1}A(v, u, b_1^{n-2}) = x$

for all $x, y, z, u, v \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q.

$b)$ By Def. 1.1 from Chapter I, by (o), by (a) and by (b), we conclude that for all $x, y, z, u, v \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following series of implications holds

---

[1] ^{-1}A is a left division in (Q, A).

$A(A(x,y,a_1^{n-2}),z,b_1^{n-2}) = A(x,A(y,a_1^{n-2},z),b_1^{n-2}) \overset{(o)}{\Rightarrow}$

${}^{-1}\!A(A(x,A(y,a_1^{n-2},z),b_1^{n-2}),z,b_1^{n-2}) = A(x,y,a_1^{n-2}) \overset{(a)}{\Rightarrow}$

${}^{-1}\!A(A(x,u,b_1^{n-2}),z,b_1^{n-2}) = A(x,{}^{-1}\!A(u,a_1^{n-2},z),a_1^{n-2}) \overset{(b)}{\Rightarrow}$

${}^{-1}\!A(v,z,b_1^{n-2}) = A({}^{-1}\!A(v,u,b_1^{n-2}),{}^{-1}\!A(u,a_1^{n-2},z),a_1^{n-2}) \overset{(o)}{\Rightarrow}$

${}^{-1}\!A(v,u,b_1^{n-2}) = {}^{-1}\!A({}^{-1}\!A(v,z,b_1^{n-2}),{}^{-1}\!A(u,a_1^{n-2},z),a_1^{n-2}).$

Whence, by the substitution $B = {}^{-1}\!A$, we conclude that the following law

$$B(B(x,z,b_1^{n-2}),B(y,a_1^{n-2},z),a_1^{n-2}) = B(x,y,b_1^{n-2})$$

holds in the $n-$groupoid (Q,B).

The proof of (ii) :

$\overline{a}$) By Th. 1.3 from Chapter III and by (o), we have

$(c)\quad (c_1^{n-2},u)^{-1} = {}^{-1}\!A({}^{-1}\!A(z,c_1^{n-2},z),c_1^{n-2},u)$

for all $u,z\in Q$ and for every sequence c_1^{n-2} over Q.

[(α) $A(\mathbf{e}(c_1^{n-2}),c_1^{n-2},z) \overset{1.3III}{=} z \overset{(o)}{\Leftrightarrow} \mathbf{e}(c_1^{n-2}) = {}^{-1}\!A(z,c_1^{n-2},z)$;

(β) $A((c_1^{n-2},u)^{-1},c_1^{n-2},u) \overset{1.3III}{=} \mathbf{e}(c_1^{n-2}) \overset{(o)}{\Leftrightarrow} {}^{-1}\!A(\mathbf{e}(c_1^{n-2}),c_1^{n-2},u) = (c_1^{n-2},u)^{-1} \overset{(\alpha)}{\Leftrightarrow}$
${}^{-1}\!A({}^{-1}\!A(z,c_1^{n-2},z),c_1^{n-2},u) = (c_1^{n-2},u)^{-1}$.]

$\overline{b}$) By Def. 1.1 from Chapter I, by Th. 1.3 from Chapter III and by (o), we conclude that for all $a,b,u,v,x\in Q$ and for every sequence a_1^{n-2} over Q the following series of equivalences holds

${}^{-1}\!A(a,c_1^{n-2},x) = b \overset{(o)}{\Leftrightarrow}$

$A(b,c_1^{n-2},x) = a \overset{1.1I}{\Leftrightarrow}$

$A((c_1^{n-2},b)^{-1},c_1^{n-2},A(b,c_1^{n-2},x)) = A((c_1^{n-2},b)^{-1},c_1^{n-2},a) \overset{1.3III}{\Leftrightarrow}$

$x = A((c_1^{n-2},b)^{-1},c_1^{n-2},a) \overset{1.1I}{\Leftrightarrow}$

$A(x,c_1^{n-2},(c_1^{n-2},a)^{-1}) = A(A((c_1^{n-2},b)^{-1},c_1^{n-2},a),c_1^{n-2},(c_1^{n-2},a)^{-1}) \overset{1.3III}{\Leftrightarrow}$

$A(x,c_1^{n-2},(c_1^{n-2},a)^{-1}) = (c_1^{n-2},b)^{-1} \overset{(o)}{\Leftrightarrow}$

${}^{-1}\!A((c_1^{n-2},b)^{-1},c_1^{n-2},(c_1^{n-2},a)^{-1}) = x \overset{(c)}{\Leftrightarrow}$

${}^{-1}\!A({}^{-1}\!A({}^{-1}\!A(u,c_1^{n-2},u),c_1^{n-2},b),c_1^{n-2},{}^{-1}\!A({}^{-1}\!A(v,c_1^{n-2},v),c_1^{n-2},a)) = x,$

i.e.

${}^{-1}\!A(a,c_1^{n-2},x) = b \Leftrightarrow$

$^{-1}\!A(^{-1}\!A(^{-1}\!A(u,c_1^{n-2},u),c_1^{n-2},b),c_1^{n-2},{}^{-1}\!A(^{-1}\!A(v,c_1^{n-2},v),c_1^{n-2},a))=x$.
Whence, by the substitution $B={}^{-1}A$, we conclude that
$B(a,c_1^{n-2},B(B(B(u,c_1^{n-2},u),c_1^{n-2},b),c_1^{n-2},B(B(v,c_1^{n-2},v),c_1^{n-2},a)))=b$
holds in the $n-$groupoid (Q,B). $\square$

1.2. Theorem *[Ušan 1997/4]: Let $n\geq 2$ and let (Q,B) be an $n-$groupoid. Let also the following laws*
(i) $B(B(x,z,b_1^{n-2}),B(y,a_1^{n-2},z),a_1^{n-2})=B(x,y,b_1^{n-2})$ *and*
(ii) $B(a,c_1^{n-2},B(B(B(u,c_1^{n-2},u),c_1^{n-2},b),c_1^{n-2},B(B(v,c_1^{n-2},v),c_1^{n-2},a)))=b$
hold in the $n-$groupoid (Q,B). Then, there is an $n-$group (Q,A) such that $^{-1}\!A=B$.

Proof. By (ii), we conclude that the following statement holds:

1° For every $a_1^n\in Q$ there is **at least one** $x\in Q$ such that $B(a_1^{n-1},x)=a_n$.

Furthermore, the following statements hold.

2° $(\forall a\in Q)(\forall z\in Q)(\forall c_i\in Q)_1^{n-2}B(a,B(z,c_1^{n-2},z),c_1^{n-2})=a$.

$\widehat{3}^\circ$ For every $x,y,a_1^{n-1}\in Q$ the following implication holds
$$B(x,a_1^{n-1})=B(y,a_1^{n-1})\Rightarrow x=y.$$

3° For every $a_1^n\in Q$ there is **exactly one** $x\in Q$ such that $B(x,a_1^{n-1})=a_n$.

4° There exists an $n-$ary operation ^{-1}B in Q such that for all $x,y\in Q$ and for every sequence a_1^{n-1} over Q

$(\overline{o})$ $$^{-1}B(x,a_1^{n-1})=y\Leftrightarrow B(y,a_1^{n-1})=x.$$

$\widehat{5}^\circ$ For every $x,y,a_1^{n-1}\in Q$ the following implication holds
$$^{-1}B(x,a_1^{n-1})={}^{-1}B(y,a_1^{n-1})\Rightarrow x=y.$$

5° For every $a_1^n\in Q$ there is **exactly one** $y\in Q$ such that $^{-1}B(y,a_1^{n-1})=a_n$.

6° For every $a_1^n\in Q$ there is **at least one** $x\in Q$ such that the following equality holds $^{-1}B(a_1^{n-1},x)=a_n$.

7° The $<1,2>-$associative law holds in $(Q,{}^{-1}B)$.

8° $(Q,{}^{-1}B)$ is an $n-$semigroup.

The proof of 2° :

1) $n \geq 3$: Putting $z = y$ in (i), we obtain

(a) $B(B(x,y,b_1^{n-2}), B(y,a_1^{n-2},y), a_1^{n-2}) = B(x,y,b_1^{n-2})$.

By 1°, we have

(b) $(\forall x \in Q)(\forall y \in Q)(\forall a \in Q)(\forall b_i \in Q)_1^{n-3}(\exists b_{n-2} \in Q)B(x,y,b_1^{n-2}) = a$.

Finally, by (a) and by (b), we conclude that the statement 2° for $n \geq 3$ holds.

2) $n = 2$: For $n = 2$ (i) is reduced to

$(\widehat{i})$ $B(B(x,z), B(y,z)) = B(x,y)$.

Putting $z = y$ in $(\widehat{i})$, we obtain

$(\widehat{a})$ $B(B(x,y), B(y,y)) = B(x,y)$.

By 1°, we have

$(\widehat{b})$ $(\forall x \in Q)(\forall a \in Q)(\exists y \in Q)B(x,y) = a$.

By $(\widehat{a})$ and $(\widehat{b})$, we obtain

$(\widehat{c})$ $(\forall x \in Q)(\forall a \in Q)(\exists y \in Q)B(a, B(y,y)) = a$.

In addition, by 1°, we have

$(\widehat{d})$ $(\forall y \in Q)(\forall u \in Q)(\exists c \in Q)y \overset{1^\circ}{=} B(u,c)$.

Whence, by $(\widehat{i})$, we obtain

$$B(y,y) \overset{(\widehat{d})}{=} B(B(u,c), B(u,c)) \overset{(\widehat{i})}{=} B(u,u),$$

i.e.

$(\widehat{e})$ $B(y,y) = B(u,u)$

for all $y, u \in Q$.

Finally, by $(\widehat{c})$ and by $(\widehat{e})$, we obtain

$$(\forall a \in Q)(\forall u \in Q)a = B(a, B(u,u)).$$

Sketch of the proof of $\widehat{3}^\circ$:

$B(x,a,b_1^{n-2}) = B(\overline{x},a,b_1^{n-2}) \Rightarrow$
$B(B(x,a,b_1^{n-2}), B(B(v,b_1^{n-2},v), a_1^{n-2}, a), a_1^{n-2}) =$
$B(B(\overline{x},a,b_1^{n-2}), B(B(v,b_1^{n-2},v), a_1^{n-2}, a), a_1^{n-2}) \overset{(i)}{\Rightarrow}$
$B(x, B(v,b_1^{n-2},v), b_1^{n-2}) = B(\overline{x}, B(v,b_1^{n-2},v), b_1^{n-2}) \overset{2^\circ}{\Rightarrow}$

$x = \overline{x}$. *[*In (i) : $a = z,\ B(v, b_1^{n-2}, v) = y$.*]*

Sketch of the proof of 3° :

$B(x, a, b_1^{n-2}) = b \overset{\widehat{3}^\circ}{\Longleftrightarrow}$
$B(B(x, a, b_1^{n-2}), B(B(v, b_1^{n-2}, v), a_1^{n-2}, a), a_1^{n-2}) =$
$B(b, B(B(v, b_1^{n-2}, v), a_1^{n-2}, a), a_1^{n-2}) \overset{(i)}{\Longleftrightarrow}$
$B(x, B(v, b_1^{n-2}, v), b_1^{n-2}) = B(b, B(B(v, b_1^{n-2}, v), a_1^{n-2}, a), a_1^{n-2}) \overset{2^\circ}{\Longleftrightarrow}$
$x = B(b, B(B(v, b_1^{n-2}, v), a_1^{n-2}, a), a_1^{n-2})$,
i.e.
$B(x, a, b_1^{n-2}) = b \Leftrightarrow x = B(b, B(B(v, b_1^{n-2}, v), a_1^{n-2}, a), a_1^{n-2})$,
whence, by $\widehat{3}^\circ$, we conclude that the statement 3° holds.

The proof of 4° : by 3°.

The proof of $\widehat{5}^\circ$:

By $(\overline{o})$, we obtain

$^{-1}B(x, a_1^{n-1}) = u \Leftrightarrow B(u, a_1^{n-1}) = x$ and
$^{-1}B(y, a_1^{n-1}) = v \Leftrightarrow B(v, a_1^{n-1}) = y$

for all $x, y, u, v, a_1^{n-1} \in Q$.
Whence, we have

$^{-1}B(x, a_1^{n-1}) = ^{-1}B(y, a_1^{n-1}) \Rightarrow x = y$

for all $x, y, a_1^{n-1} \in Q$.

The proof of 5° :

By $(\overline{o})$, we obtain

$^{-1}B(x, a_1^{n-1}) = b \Leftrightarrow x = B(b, a_1^{n-1})$

for all $x, b, a_1^{n-1} \in Q$. Whence, by $\widehat{5}^\circ$, we conclude that the statement 5° holds.

The proof of 6° :

By 4°, we obtain

$^{-1}B(a, a_1^{n-2}, x) = b \Leftrightarrow B(b, a_1^{n-2}, x) = a$.

for all $a, b, x \in Q$ and for every sequence a_1^{n-2} over Q. Whence, by 1°, we conclude that the statement 6° holds.

Sketch of the proof of 7° :

a) By $(\bar{o})$, we have

(a) $B(v,u,b_1^{n-2}) = x \Leftrightarrow {}^{-1}B(x,u,b_1^{n-2}) = v$ and

(b) $B(u,a_1^{n-2},z) = y \Leftrightarrow {}^{-1}B(y,a_1^{n-2},z) = u$

for all $x,y,z,u,v \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q.

b) The following series of implications holds

$$B(u,v,b_1^{n-2}) = B(B(v,z,b_1^{n-2}), B(u,a_1^{n-2},z), a_1^{n-2}) \overset{(\bar{o})}{\Longrightarrow}$$
$$B(v,z,b_1^{n-2}) = {}^{-1}B(B(u,v,b_1^{n-2}), B(u,a_1^{n-2},z), a_1^{n-2}) \overset{(a)}{\Longrightarrow}$$
$$B({}^{-1}B(x,u,b_1^{n-2}), z, b_1^{n-2}) = {}^{-1}B(x, B(u,a_1^{n-2},z), a_1^{n-2}) \overset{(b)}{\Longrightarrow}$$
$$B({}^{-1}B(x, {}^{-1}B(y,a_1^{n-2},z), b_1^{n-2}), z, b_1^{n-2}) = {}^{-1}B(x,y,a_1^{n-2}) \overset{(\bar{o})}{\Longrightarrow}$$
$${}^{-1}B({}^{-1}B(x,y,a_1^{n-2}), z, b_1^{n-2}) = {}^{-1}B(x, {}^{-1}B(y,a_1^{n-2},z), b_1^{n-2}).$$

The proof of 8° :

a) For $n = 2$, by 7°. b) For $n \geq 3$, by 7° and by Prop. 2.1 from Chapter III.

Finally, by $1°, 5°, 8°$ and by Th. 3.1 (or Th. 3.2) from Chapter III, we conclude that Th. 1.2 holds. □

1.3. Remark: *Similarly, the $n-$group (Q,A) can be described by the $n-$groupoid (Q,A^{-1}). [$A^{-1}(x,a_1^{n-2},y) = z \overset{def}{\Longleftrightarrow} A(x,a_1^{n-2},z) = y$; A^{-1} is a right division in $n-$group (Q,A).]*

2 One proposition of the $n-$subgroups

2.1. Theorem: *Let (Q,A) be an $n-$group, ${}^{-1}$ its inverse operation and $n \geq 2$. Also, let $H \in P(Q)\backslash\{\emptyset\}$. Then (H,A) is an $n-$subgroup of the $n-$group (Q,A) iff for all $x,y \in H$ and for every sequence a_1^{n-2} over H the following statement holds*

$$A(x,a_1^{n-2},(a_1^{n-2},y)^{-1}) \in H.$$

Proof. By Th. 1.1, by Th. 1.2 and by Prop. 2.1 from VIII *[*$:{}^{-1}A(x,a_1^{n-2},y) = A(x,a_1^{n-2},(a_1^{n-2},y)^{-1})$*]*. □

2.2. Remark: *Th. 2.1 for $n = 2$ is a well known proposition.*

3 $TS - n-$groups as $n-$groupoids with laws

3.1. Definition: *Let (Q, A) be an $n-$quasigroup and $n \geq 2$. Also let α be a permutation in the set $\{1, 2, \ldots, n+1\}$. Moreover, let*

$$A^{\alpha}(x_1^n) = a_{n+1} \overset{def}{\Leftrightarrow} A(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = x_{\alpha(n+1)}$$

for all $x_1^{n+1} \in Q$. We say that (Q, A) is a **totally symmetric $n-$quasigroup** *[briefly: TS-n-quasigroup] iff for any permutation α on $\{1, 2, \ldots, n+1\}$ we have $A^{\alpha} = A$. In the case when $\alpha = (1, n+1)$ instead of A^{α} we have ${}^{-1}A$. Similarly in the case $\alpha = (n, n+1)$ instead of A^{α} we write A^{-1}.*

3.2. Proposition *[Ušan 1999/6]: Let $n \geq 2$ and let (Q, B) be a $TS -$ $n-$group. Then the following laws*

(i) $B(B(x, z, b_1^{n-2}), B(y, a_1^{n-2}, z), a_1^{n-2}) = B(x, y, b_1^{n-2})$,

(ii) $B(a, c_1^{n-2}, B(B(B(u, c_1^{n-2}, u), c_1^{n-2}, b), c_1^{n-2}, B(B(v, c_1^{n-2}, v), c_1^{n-2}, v), c_1^{n-2}, a))) =$ b,

(iii) $B(x, a_1^{n-2}, y) = B(x, a_1^{n-2}, B(B(y, a_1^{n-2}, y), a_1^{n-2}, y)) = b$ *and*

(iv) $B(x, y, a_1^{n-2}) = B(y, x, a_1^{n-2})$

hold in the $n-$groupoid (Q, B).

Proof. *a) By ${}^{-1}B = B$ (Def. 3.1), and by Th. 1.1, we have (i) and (ii).*
b) By ${}^{-1}B = B$ and by Prop. $2.1 - (4_R)$ from Chapter VIII [:$A(x, a_1^{n-2}, y) =$ ${}^{-1}A(x, a_1^{n-2}, {}^{-1}A({}^{-1}A(y, a_1^{n-2}, y), a_1^{n-2}, y)$], we obtain (iii). c) By Def.3.1, we have also (iv). □

3.3. Theorem: *[Ušan 1999/6]: Let $n \geq 2$ and let (Q, B) be an $n-$groupoid. Let also the laws $(i) - (iv)$ hold. Then (Q, B) is a TS-n-group.*

Proof. Firstly we observe that under the assumption the following statements hold:

$^\circ 1$ There is an $n-$group (Q, A) such that ${}^{-1}\!A = B$;

$^\circ 2$ ${}^{-1}\!B = A$;

$^\circ 3$ ${}^{-1}\!B = B$;

$^\circ 4$ (Q, B) is an $n-$group;

$^\circ 5$ For all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds $(a_1^{n-2}, x)^{-1} = x$, where ${}^{-1}$ is an inverse operation in the $n-$group (Q, B);

$^\circ 6$ $B^{-1} = B$;

$^\circ 7$ For all $x, y \in Q$ and for every sequence a_1^{n-2} over Q the equality $B(x, a_1^{n-2}, y) = B(y, a_1^{n-2}, x)$ holds;

$^\circ 8$ Let $n \geq 3$. Then, for all $x_1^n \in Q$ and for all permutation α on $\{1, \ldots, n\}$ the following equality holds

$$B(x_1^n) = B(x_{\alpha(1)}, \ldots, x_{\alpha(n)}); \text{ and}$$

$^\circ 9$ Let $n \geq 3$. Then, for all $x_1^n \in Q$ and for all $i \in \{2, \ldots, n-1\}$ the equality $B^{(i)} = B$ holds, where

(E) $\quad B^{(i)}(a_1^{i-1}, x, a_i^{n-1}) = y \overset{def}{\Leftrightarrow} B(a_1^{i-1}, y, a_i^{n-1}) = x.$

The proof of $^\circ 1$: By $(i), (ii)$ and by Th. 1.2.

The proof of $^\circ 2$: ${}^{-1}\!B \overset{^\circ 1}{=} {}^{-1}({}^{-1}\!A) = A$.

The proof of $^\circ 3$:

By $^\circ 1, ^\circ 2$ and by Prop. 2.1 $- (4_l)$ from Chapter VIII, we have

$(\overline{iii})$ ${}^{-1}\!B(x, a_1^{n-2}, y) = B(x, a_1^{n-2}, B(B(y, a_1^{n-2}, y), a_1^{n-2}, y))$

for all $x, y \in Q$ and for every sequence a_1^{n-2} over Q. By (iii) and by $(\overline{iii})$, we obtain ${}^{-1}\!B = B$.

The proof of $^\circ 4$:

Firstly, by $^\circ 2$ and $^\circ 3$, we have $A = B$. Whence, by $^\circ 1$, we conclude that $^\circ 4$ holds.

The proof of °5 :

By °3,° 4 and by Prop. 2.1 – (1_l) from Chapter VIII, we have

$$B(x, a_1^{n-2}, y) = B(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1})$$

for all $x, y \in Q$ and for every sequence a_1^{n-2} over Q. Whence, by Def. 1.1 from Chapter I, we obtain

$$(\forall y \in Q)(\forall a_i \in Q)_1^{n-2} (a_1^{n-2}, y)^{-1} = y.$$

The proof of °6 :

By °4,° 5 and by Prop. 2.1 – (1_r) from Chapter VIII.

The proof of °7 :

Firstly:

$$(R) \quad B^{-1}(x, a_1^{n-2}, y) = z \overset{def}{\Longleftrightarrow} B(x, a_1^{n-2}, z) = y$$

for all $x, y, z \in Q$ and for every sequence a_1^{n-2} over Q.

Further:

$$\begin{aligned} B(x, a_1^{n-2}, y) = z \quad & \overset{\circ 3}{\Longleftrightarrow} {}^{-1}B(x, a_1^{n-2}, y) = z \\ & \overset{(0)}{\Longleftrightarrow} B(z, a_1^{n-2}, y) = x \\ & \overset{\circ 6}{\Longleftrightarrow} B^{-1}(z, a_1^{n-2}, y) = x \\ & \overset{(R)}{\Longleftrightarrow} B(z, a_1^{n-2}, x) = y \\ & \overset{(0)}{\Longleftrightarrow} {}^{-1}B(y, a_1^{n-2}, x) = z \\ & \overset{\circ 3}{\Longleftrightarrow} B(y, a_1^{n-2}, x) = z. \end{aligned}$$

Remark: For $n = 2$ $B(x, a_1^{n-2}, y) = B(y, a_1^{n-2}, x)$ is the law (iv).

The proof of °8 :

Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary nHG−algebra associated to the n−group (Q, B) *[*Chapter IV-2*]*. By Th. 4.1 from Chapter IV, there is a sequence a_1^{n-2} over Q such that

$$(M) \qquad x \cdot y = A(x, a_1^{n-2}, y) \text{ and}$$

$$(A) \qquad \varphi(x) = A(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2}),$$

where $\mathbf{e}$ is a $\{1, n\}$−neutral operation of the n−group (Q, B).

By (M) and by °7, we have

$$(K) \qquad (\forall x \in Q)(\forall y \in Q) x \cdot y = y \cdot x.$$

In addition, by (A) and by (iv), we obtain

(I) $\qquad (\forall x \in Q)\varphi(x) = x.$

Finally, by (K), by (I) and by Def. 2.3 from IV, we conclude that the statement $^\circ 8$ holds.

Sketch of the proof of $^\circ 9$:

$$\begin{aligned} B^{(i)}(a_1^{i-1}, x, a_i^{n-1}) = y \quad &\overset{(E)}{\Longleftrightarrow} B(a_1^{i-1}, y, a_i^{n-1}) = x \\ &\overset{\circ 8}{\Longleftrightarrow} B(y, a_1^{i-1}, a_i^{n-1}) = x \\ &\overset{(0)}{\Longleftrightarrow} {}^{-1}B(x, a_1^{i-1}, a_i^{n-1}) = y \\ &\overset{\circ 3}{\Longleftrightarrow} B(x, a_1^{i-1}, a_i^{n-1}) = y \\ &\overset{\circ 8}{\Longleftrightarrow} B(a_1^{i-1}, x, a_i^{n-1}) = y. \end{aligned}$$

Finally: 1) By $^\circ 4, ^\circ 3, ^\circ 6$ and by $^\circ 7$, we conclude that Th.3.3 for $n = 2$ holds; and 2) By $^\circ 4, ^\circ 3, ^\circ 6, ^\circ 8$ and by $^\circ 9$, we conclude that Th. 3.3 for $n \geq 3$ also holds. /See remark in the proof of $^\circ 7$/ □

4 Remarks

4.1. A variety of groups of the type < 2 > has been considered in [Higman, Neuman 1952] /See, also [Cohn 1968] and [Kurosh 1967]/. The investigation of this paper was extended in [Tasić 1988] for groups, for rings and, more generally, for $\Omega-$groups.

4.2. In [Furnstenberg 1955] a group is described as a groupoid (Q, B) which satisfies one law (i.e. our (i) for $n = 2$) and in which the equality $B(a, x) = b$ has at least one solution x for each $a, b \in Q$.

4.3. In [Ušan, Galić 2000] a class of $(m, n)-$rings with left and right zero has been described as a variety of algebras of type $< 3m + n - 5, 0 >$. In [Sorkin 1957] rings /$(2, 2)-$rings / have been described as 3-groupoids with one law.

Chapter XIV

ON SEMIINVARIANT (INVARIANT) $n-$SUBGROUP

1 Auxiliary propositions

1.1. Proposition *[Ušan 1994]: Let (Q, A) be an $n-$group, $^{-1}$ its inverse operation and $n \geq 2$. Then for all $a, b \in Q$ and for every sequence c_1^{n-2} over Q the following equality holds*

$$(c_1^{n-2}, A(a, c_1^{n-2}, b))^{-1} = A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, (c_1^{n-2}, a)^{-1}).$$

Proof. Let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of the $n-$group (Q, A). Also, let a and b be arbitrary elements of the set Q and c_1^{n-2} an arbitrary sequence over Q. Then the following sequence of equivalences holds

$$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \overset{1.1I}{\Longleftrightarrow}$$
$$A((c_1^{n-2}, A(a, c_1^{n-2}, b))^{-1}, c_1^{n-2}, A(A(a, c_1^{n-2}, b), c_1^{n-2}, x)) =$$
$$A((c_1^{n-2}, A(a, c_1^{n-2}, b))^{-1}, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \overset{1.3III}{\Longleftrightarrow}$$
$$x = (c_1^{n-2}, A(a, c_1^{n-2}, b))^{-1},$$

and hence $(c_1^{n-2}, A(a, c_1^{n-2}, b))^{-1}$ is a solution of the equation

(α) $A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2})$

(for the unknown x). With regard to this, for every $x \in Q$ the following sequence of equivalences holds

$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \overset{1.1I}{\Longleftrightarrow}$

$A(a, c_1^{n-2}, A(b, c_1^{n-2}, x)) = \mathbf{e}(c_1^{n-2}) \overset{1.1I}{\Longleftrightarrow}$

$A((c_1^{n-2}, a)^{-1}, c_1^{n-2}, A(a, c_1^{n-2}, A(b, c_1^{n-2}, x))) =$

$A((c_1^{n-2}, a)^{-1}, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \overset{1.3III}{\Longleftrightarrow}$

$A(b, c_1^{n-2}, x) = (c_1^{n-2}, a)^{-1} \overset{1.1I}{\Longleftrightarrow}$

$A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, A(b, c_1^{n-2}, x)) = A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, (c_1^{n-2}, a)^{-1}) \overset{1.3III}{\Longleftrightarrow}$

$x = A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, (c_1^{n-2}, a)^{-1})$,
and hence we have
$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \Longleftrightarrow$
$x = A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, (c_1^{n-2}, a)^{-1})$.
Thereby we conclude that $A((c_1^{n-2}, b)^{-1}, c_1^{n-2}, (c_1^{n-2}, a)^{-1})$ is also a solution of the equation (α). □

1.2. Proposition: *Let (Q, A) be an $n-$group, $^{-1}$ its inverse operation and $n \geq 2$. Then for all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds*

$$(a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1} = x.$$

Proof. By Th. 1.3 Chapter III, for all $x \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

$$((a_1^{n-2}, x)^{-1}, a_1^{n-2}, x) = \mathbf{e}(a_1^{n-2}) \text{ and}$$
$$((a_1^{n-2}, x)^{-1}, a_1^{n-2}, (a_1^{n-2}, (a_1^{n-2}, x)^{-1})^{-1}) = \mathbf{e}(a_1^{n-2}),$$

whence, by Def. 1.1 from I, we conclude that the Prop.1.2 holds. □

1.3. Proposition: *Let (Q, A) be an $n-$group, $^{-1}$ its inverse operation and $n \geq 2$. Then for all $x, y \in Q$ and for every sequence c_1^{n-2} over Q the following equality holds*

$$A((c_1^{n-2}, x)^{-1}, c_1^{n-2}, y) = c_{n-1} \Longleftrightarrow y = A(x, c_1^{n-1}).$$

Sketch of the proof.
$A((c_1^{n-2}, x)^{-1}, c_1^{n-2}, y) = c_{n-1} \overset{1.1I}{\Longleftrightarrow}$
$A(x, c_1^{n-2}, A((c_1^{n-2}, x)^{-1}, c_1^{n-2}, y)) = A(x, c_1^{n-2}, c_{n-1}) \overset{1.1I}{\Longleftrightarrow}$
$A(A(x, c_1^{n-2}, (c_1^{n-2}, x)^{-1}), c_1^{n-2}, y) = A(x, c_1^{n-1}) \overset{1.3III}{\Longleftrightarrow}$
$A(\mathbf{e}(c_1^{n-2}), c_1^{n-2}, y) = A(x, c_1^{n-1}) \overset{1.3III}{\Longleftrightarrow}$
$y = A(x, c_1^{n-1})$. □

2 Main proposition

2.1. Definition *[Rusakov 1992]: Let* $n \geq 2$, (Q, A) *be an* $n-$*group, and* (H, A) *its* $n-$*subgroup. Then:* (a) (H, A) *is a* **semiinvariant** $n-$*subgroup of the* $n-$*group* (Q, A) *iff the following formula holds*

$$(\forall x \in Q)(\forall h_i \in H)_1^{n-2}(\forall h \in H)(\exists \overline{h} \in H)A(h, h_1^{n-2}, x) = A(x, h_1^{n-2}, \overline{h});$$

and (b) (H, A) *is an* **invariant** *(normal)* $n-$*subgroup of the* $n-$*group* (Q, A) *iff* (H, A) *is a semiinvariant* $n-$*subgroup of the* $n-$*group* (Q, A) *and for all* $x \in Q$ *the following equality holds*

$$A(x, \overset{n-1}{H}) = A(H, x, \overset{n-2}{H}).$$

2.2. Theorem: *[Ušan, Žižović 1999/1] Let* (Q, A) *be an* $n-$*group,* $^{-1}$ *its inverse operation,* (H, A) *its semiinvariant* $n-$*subgroup and* $n \geq 2$. *Also let*

(1) $x\theta y \overset{def}{\Longleftrightarrow} (\exists h_i \in H)_1^{n-2}(\exists h \in H)A((h_1^{n-2}, x)^{-1}, h_1^{n-2}, y) = h$

for all $x, y \in Q$. *Then,* θ *is a congruence on* (Q, A).

Proof. Firstly, by (H, A) is an $n-$subgroup of the $n-$group (Q, A), and by Th. 3.1 (3.4 − (a_i)) from Chapter III, we conclude that the following statements hold:

1° For every sequence h_1^{n-2} over H, $\mathbf{e}(h_1^{n-2}) \in H$ [for $n = 2 : \mathbf{e}(\emptyset) \in H$];

2° For all $h_1^{n-1} \in H$, $(h_1^{n-1})^{-1} \in H$; and

3° For all $h_1^n \in H$, $A(h_1^n) \in H$.

In addition, we observe that under the assumptions the following statements hold:

°1 θ is a reflexive relation;

°2 θ is a symmetric relation;

°3 θ is a transitive relation; and

°4 For all $a, b, x_1^{n-1} \in Q$ and for every $i \in \{1, \dots, n\}$ the following implication holds

$$a\ \theta\ b \Rightarrow A(x_1^{i-1}, a, x_i^{n-1})\theta A(x_1^{i-1}, b, x_i^{n-1}).$$

The proof of °1 :

By (1) and by Th. 1.3 from Chapter III, we have

$$x\theta x \Leftrightarrow (\exists h_i \in H)^{n-2}(\exists h \in H)\mathbf{e}(h_1^{n-2}) = h$$

for all $x \in Q$. Whence, by 1° *[* $(\forall h_i \in H)_1^{n-2}(\exists h \in H)\mathbf{e}(h_1^{n-2}) = h$*]*, we conclude that the statement °1 holds.

The proof of °2 :

By Prop. 1.1 and by Prop. 1.2, we conclude that the following series of implications holds

$$A((h_1^{n-2},x)^{-1},h_1^{n-2},y) = h_{n-1} \Rightarrow$$
$$(h_1^{n-2},A((h_1^{n-2},x)^{-1},h_1^{n-2},y))^{-1} = (h_1^{n-1})^{-1} \overset{1.1}{\Longrightarrow}$$
$$A((h_1^{n-2},y)^{-1},h_1^{n-2},(h_1^{n-2},(h_1^{n-2},x)^{-1})^{-1}) = (h_1^{n-1})^{-1} \overset{1.2}{\Longrightarrow}$$
$$A((h_1^{n-2},y)^{-1},h_1^{n-2},x) = (h_{n-1})^{-1}$$

for all $h_1^{n-1} \in H$ and for every $x, y \in Q$. Whence, by 2° and by (1), we conclude that the °2 holds.

The proof of °3 :

a) By Prop. 1.3, we have

$$A((h_1^{n-2},x)^{-1},h_1^{n-2},y) = h_{n-1} \Leftrightarrow y = A(x,h_1^{n-1}) \text{ and}$$
$$A((\overline{h}_1^{n-2},y)^{-1},\overline{h}_1^{n-2},z) = \overline{h}_{n-1} \Leftrightarrow z = A(y,\overline{h}^{n-1})$$

for all $x, y, z \in Q$, for every sequence h_1^{n-1} over H and for every sequence $\overline{h}_{n-1}$ over H.

b) The following series of implications also holds

$$y = A(x,h_1^{n-1}) \wedge z = A(y,\overline{h}_1^{n-1}) \Rightarrow$$
$$z = A(A(x,h_1^{n-1}),\overline{h}_1^{n-1}) \Rightarrow$$
$$z = A(x,h_1^{n-2},A(h_{n-1},\overline{h}_1^{n-1}))$$

for all $x, y, z \in Q$ and for every $h_1^{n-1}, \overline{h}_1^{n-1} \in H$.

Finally, by a), b), 3° and by (1), we conclude that °3 holds.

The proof of °4 :

Firstly, by Prop. 1.3 and by (1), we obtain

$(\overline{1})$ $a\theta b \Leftrightarrow (\exists h_i \in H)_1^{n-1} b = A(a,h_1^{n-1})$

for all $a, b \in Q$.

In addition, by $(\overline{1})$ and by Def. $2.1-(a)$, we conclude that the following series of equalities holds

$$\begin{aligned} A(x_1^{i-1}, b, x_1^{n-1}) &\overset{(\overline{1})}{=} A(x_1^{i-1}, A(a, h_1^{n-1}), x_i^{n-1}) \\ &\overset{1.1I}{=} A(x_1^{i-1}, a, A(h_1^{n-1}, x_i), x_{i+1}^{n-1}) \\ &\overset{2.1}{=} A(x_1^{i-1}, a, A(x_i, h_2^{n-1}, \overline{h}_1), x_{i+1}^{n-1}) \\ &- \\ &- \\ &= A(A(x_1^{i-1}, a, x_i^{n-1}), h_{n-i+1}^{n-1}, \overline{h}_{n-1}, \dots, \overline{h}_1). \end{aligned}$$

Whence, by $(\overline{1})$ and by $°2$, we conclude that for all $a, b \in Q$, for all $i \in \{1, \dots, n\}$ and for every $x_1^{n-1} \in Q$ the following implication holds

$$a\theta b \Rightarrow A(x_1^{i-1}, a, x_i^{n-1})\theta A(x_1^{i-1}, b, x_i^{n-1}).$$

Finally, by $°1 -° 4$, we conclude that Th. 2.2 holds. □

2.3. Remark: *For $n \geq 3$ there exists an $n-$group (Q, A) together with its congruence relation θ, such that for all $C_t \in Q/\theta$ the following statement holds: (C_t, A) is not an $n-$subgroup of the $n-$ group (Q, A). Cf. 5.1-5.4 from Chapter VI.*

Chapter XV

SOME OTHER PROPERTIES OF $n-$GROUPS AND CLOSE OPERATIONS

1 On power in $n-$groups

1.1. Definition *[Ušan 1999/2]: Let $n \geq 2$. Let (Q, A) be an $n-$group,* $\mathbf{e}$ *its $\{1, n\}-$neutral operation and $^{-1}$ its inverse operation. Then, we say that a^m $(m \in Z)$ is the $m-$th power of the element a in (Q, A) iff*

(1) $a^1 \overset{def}{=} a$;

(2) $a^{k+1} \overset{def}{=} A(a^k, \overset{n-2}{a}, a), k \geq 1$;

(3) $a^\circ \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$; *and*

(4) $a^{-k} \overset{def}{=} (\overset{n-2}{a}, a^k)^{-1}, k \geq 1$.

1.2. Remark*: For $n = 2$, the conditions (1)-(4) reduce to the conditions*

($\hat{1}$) $a^1 \overset{def}{=} a$;

($\hat{2}$) $a^{k+1} \overset{def}{=} A(a^k, a), k \geq 1$;

($\hat{3}$) $a^\circ \overset{def}{=} e[= \mathbf{e}(\emptyset)]$; *and*

($\hat{4}$) $a^{-k} \overset{def}{=} (a^k)^{-1}, k \geq 1$.

Let $n \geq 3$, (Q, A) be an $n-$group, $^{-1}$ its inverse operation and $\mathbf{e}$ its $\{1, n\}-$neutral operation. Let also a be an arbitrary element of the set Q and for all $x, y \in Q$ let

(5) $x \square y \overset{def}{=} A(x, \overset{n-2}{a}, y)$,

(6) $x^{-\mathbf{1}} \overset{def}{=} (\overset{n-2}{a}, x)^{-1}$ and

(7) $e_\Box \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$.

Then, $(Q, \Box)$ is a group with the inverse operation $^{\mathbf{-1}}$ and the neutral element $e_\Box$. By the convention (5)-(7), the conditions (1)-(4) can be formulated in the following way:

($\bar{1}$) $a^1 = a$;

($\bar{2}$) $a^{k+1} = a^k \Box a,\ k \geq 1$;

($\bar{3}$) $a^\circ = e_\Box$;and

($\bar{4}$) $a^{-k} = (a^k)^{\mathbf{-1}},\ k \geq 1$.

Hence, the following proposition is fulfilled:

1.3. Theorem *[Ušan 1999/2]: Let $n \geq 3$, $(Q, \{A, ^{-1}, \mathbf{e}\})$ be an $n-$group as algebra of the type $< n, n-1, n-2 >$ [Chapter III], a be an arbitrary element from Q and $(Q, \{\Box, ^{\mathbf{-1}}, e_\Box\})$ the group defined by (5)-(7). Let, also, Z be the set of all integers. Then: a^m $(m \in Z)$ is the $m-$th power of the element a in the $n-$group $(Q, \{A, ^{-1}, \mathbf{e}\})$ iff a^m is the $m-$th power of a in the group $(Q, \{\Box, ^{\mathbf{-1}}, e_\Box\})$.*

1.4. Theorem *[Ušan 1999/2]: Let $n \geq 2$, $(Q, \{A, ^{-1}, \mathbf{e}\})$ be an $n-$group as an algebra of the type $< n, n-1, n-2 >$, a be an arbitrary element from Q. Let, also, Z be the set of all integers. Then for every $\alpha, \alpha_1, \ldots, \alpha_n \in Z$ the following equalities hold*

(8) $A(a^{\alpha_1}, \ldots, a^{\alpha_n}) = a^{\sum_{i=1}^{n} \alpha_i + 2 - n}$

(9) $(a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}, a^{\alpha})^{-1} = a^{-\alpha - 2(\sum_{i=1}^{n-2} \alpha_i + 2 - n)}$

(10) $\mathbf{e}(a^{\alpha_1}, \ldots, a^{\alpha_{n-2}}) = a^{-\sum_{i=1}^{n-2} \alpha_i + n - 2}$.

Proof. 1) Let a be an arbitrary element of the set Q and for all $x, y \in Q$ let

$x \Box y \overset{def}{=} A(x, \overset{n-2}{a}, y)$; *[(5)]*;

(11) $\varphi_\Box(x) \overset{def}{=} A(\mathbf{e}(\overset{n-2}{a}), x, \overset{n-2}{a})$; and

(12) $b_\square \overset{def}{=} A(\overbrace{\mathbf{e}(\overset{n-2}{a})}^{n}|)$.

Whence, by Th. 3.1 from Chapter IV, we conclude that the $(Q, \{\square, \varphi_\square, b_\square\})$ is an $nHG-$algebra associated to the $n-$group (Q, A).

2) We prove that under the assumption the following statements hold

1° For every $m \in Z$ we have $\varphi_\square(a^m) = a^m$;

2° $b_\square = a^{-(n-2)}$; and

3° For all $a, x \in Q$ the following equality holds

$$(\overset{n-2}{a}, x)^{-1} = ((\varphi_\square(a)\square \ldots \square\varphi_\square^{n-2}(a)\square b_\square)\square x\square(\varphi_\square(a)\square \ldots \square\varphi_\square^{n-2}(a)\square b_\square))^{-\mathbf{1}}.$$

The proof of 1° :

$a)$ $m = 1$:

$$\begin{aligned}\varphi_\square(a^1) &\overset{(1)}{=} \varphi_\square(a) \overset{(11)}{=} A(\mathbf{e}(\overset{n-2}{a}), a, \overset{n-2}{a})\\ &= A(\mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}, a)\\ &\overset{1.3III}{=} a \overset{(1)}{=} a^1.\end{aligned}$$

$b)$ $m = k \geq 2$:

$$\begin{aligned}\varphi_\square(a^k) &\overset{(11)}{=} A(\mathbf{e}(\overset{n-2}{a}), a^k, \overset{n-2}{a})\\ &\overset{(1)(2)}{=} A(\mathbf{e}(\overset{n-2}{a}), \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}), \overset{n-2}{a})\\ &\overset{6.3VI}{=} A(\mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}, \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}))\\ &\overset{1.3III}{=} \overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a})\\ &\overset{(1)(2)}{=} a^k.\end{aligned}$$

$c)$ $m = 0$:

$$\begin{aligned}\varphi_\square(a^\circ) &\overset{(3)}{=} \varphi_\square(\mathbf{e}(\overset{n-2}{a}))\\ &\overset{(11)}{=} A(\mathbf{e}(\overset{n-2}{a}), \mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a})\\ &\overset{1.1IV}{=} \mathbf{e}(\overset{n-2}{a}) \overset{(3)}{=} a^\circ.\end{aligned}$$

$d)$ $m = -k,\ k \geq 1$:

$$
\begin{aligned}
\varphi_{\Box}(a^{-k}) &\overset{(4)}{=} A(\mathbf{e}(\overset{n-2}{a}),(\overset{n-2}{a},a^k)^{-1},\overset{n-2}{a})\\
&\overset{1.3III}{=} A(A((\overset{n-2}{a},a^k)^{-1},\overset{n-2}{a},a^k),(\overset{n-2}{a},a^k)^{-1},\overset{n-2}{a})\\
&\overset{1.1I}{=} A((\overset{n-2}{a},a^k)^{-1},A(\overset{n-2}{a},a^k,(\overset{n-2}{a},a^k)^{-1}),\overset{n-2}{a})\\
&\overset{(1)(2)}{=} A((\overset{n-2}{a},a^k)^{-1},A(\overset{n-2}{a},\overset{k-1}{A},(\overset{(k-1)(n-1)+1}{a}),(\overset{n-2}{a},a^k)^{-1}),\overset{n-2}{a})\\
&\overset{6.3VI}{=} A((\overset{n-2}{a},a^k)^{-1},A(\overset{k-1}{A}(\overset{(k-1)(n-1)+1}{a}),\overset{n-2}{a},(\overset{n-2}{a},a^k)^{-1}),\overset{n-2}{a})\\
&\overset{(1)(2)}{=} A((\overset{n-2}{a},a^k)^{-1},A(a^k,\overset{n-2}{a},(\overset{n-2}{a},a^k)^{-1}),\overset{n-2}{a})\\
&\overset{1.3III}{=} A((\overset{n-2}{a},a^k)^{-1},\mathbf{e}(\overset{n-2}{a}),\overset{n-2}{a})\\
&\overset{1.1IV}{=} (\overset{n-2}{a},a^k)^{-1}\overset{(4)}{=} a^{-k}.
\end{aligned}
$$

The proof of 2° :

$$
\begin{aligned}
a\Box a &\overset{(5)}{=} A(a,\overset{n-2}{a},a)\\
&\overset{1)}{=} a\Box\varphi_{\Box}(a)\Box\ldots\Box\varphi_{\Box}^{n-1}(a)\Box b_{\Box}\\
&\overset{(1)}{=} a\Box\varphi_{\Box}(a^1)\Box\ldots\Box\varphi_{\Box}^{n-1}(a^1)\Box b_{\Box}\\
&\overset{1^\circ}{=} a\Box a\Box\ldots\Box a\Box b_{\Box},
\end{aligned}
$$

whence, by Th.1.3, we conclude that

$$b_{\Box} = a^{-(n-2)}.$$

The proof of 3° :

$$A((\overset{n-2}{a},x)^{-1},\overset{n-2}{a},x) = \mathbf{e}(\overset{n-2}{a}) \overset{2.3IV}{\Longrightarrow}$$
$$(\overset{n-2}{a},x)^{-1}\Box\varphi_{\Box}(a)\Box\ldots\Box\varphi_{\Box}^{n-2}(a)\Box b_{\Box}\Box x = \mathbf{e}(\overset{n-2}{a}) \overset{4.2IV}{\Longrightarrow}$$
$$(\overset{n-2}{a},x)^{-1}\Box\varphi_{\Box}(a)\Box\ldots\Box\varphi_{\Box}^{n-2}(a)b\Box x = (\varphi_{\Box}(a)\Box\ldots\Box\varphi_{\Box}^{n-2}(a)\Box b_{\Box})^{-\mathbf{1}},$$

whence, we conclude that the statement 3° holds.

3) Finally, by $1^\circ - 3^\circ$, and by Th. 1.3, we conclude that for every $\alpha,\alpha_1,\ldots,\alpha_n \in Z$ the following equalities hold

$$
\begin{aligned}
A(a^{\alpha_1},\ldots,a^{\alpha_n}) &= a^{\alpha_1}\Box\ldots\Box a^{\alpha_n}\Box a^{-(n-2)} \\
&= a^{\sum\limits_{i=1}^{n}\alpha_i+2-n}, \\
(a^{\alpha_1},\ldots,a^{\alpha_{n-2}},a^{\alpha})^{-1} &\overset{3^\circ}{=}(a^{\alpha_1}\Box\ldots\Box a^{\alpha_{n-2}}\Box a^{-(n-2)}\Box a^{\alpha}\Box a^{\alpha_1}\Box\ldots\Box a^{\alpha_{n-2}}\Box \\
&\quad a^{-(n-2)})^{-\mathbf{1}} = a^{-\alpha-2\cdot(\sum\limits_{i=1}^{n-2}\alpha_i-(n-2))}, \text{ and} \\
\mathbf{e}(a^{\alpha_1},\ldots,a^{\alpha_{n-2}}) &\overset{4.2IV}{=}(a^{\alpha_1}\Box\ldots\Box a^{\alpha_{n-2}}\Box a^{-(n-2)})^{-\mathbf{1}} \\
&= a^{-\sum\limits_{i=1}^{n-2}\alpha_i+n-2}. \quad \Box
\end{aligned}
$$

1.5. Remark: *For $n=2$, the equality (8) reduces to the well-known equality*

$$A(a^{\alpha_1},a^{\alpha_2})=a^{\alpha_1+\alpha_2}.$$

Moreover, for $n=2$, by convection $\sum\limits_{i=1}^{0}\alpha_i\overset{def}{=}0$, the equalities (9) and (10) reduce to the well-known equalities

$$(a^{\alpha})^{-1}=a^{-\alpha} \text{ and } \mathbf{e}(\emptyset)=a^{\circ},$$

where $\mathbf{e}(\emptyset)$ is a neutral element of the group (Q,A) and $\alpha\in Z$.

1.6. Remarks: *a) Definition of the $s-$***th** *$n-$***adic power** *[Post 1940][1]: Let $n\geq 3$ and let (Q,A) be an $n-$group. Let, also, Z be the set of all integers. Then we say that $a^{<s>}$ $(s\in Z)$ is the $s-$th $n-$adic power of the element a in (Q,A) iff*

(a) $a^{<s>}\overset{def}{=}a,\ s=0;$

(b) $a^{<s>}\overset{def}{=}\overset{s}{A}(\overset{s(n-1)+1}{a}),\ s>0;$ *and*

(c) $a^{<s>}\overset{def}{=}x,\ s<0,$ *where* $\overset{-s}{A}(x,\overset{-s(n-1)}{a})=a.$

b) [Ušan 1999/2]: $a^{<s>}=a^{s+1}$ for all $s\in Z$. c) [Post 1940]: $A(a^{<s_1>},\ldots,a^{<s_n>})=a^{<s_1+\ldots+s_n+1>}$ for all $s_1^n\in Z$.

The proof of b) :

$\overline{1}$ $a^{<0>}\overset{(a)}{=}a\overset{(1)}{=}a^1.$

$\overline{2}$ $s>0:\quad a^{<s>}\overset{(b)}{=}\overset{s}{A}(\overset{s(n-1)+1}{a})\overset{(1)(2)}{=}a^{s+1}.$

$\overline{3}$ $s=-1:\quad A(a^{<-1>},\overset{n-1}{a})\overset{(c)}{=}a \Leftrightarrow A(a^{<-1>},\overset{n-2}{a},a)=a,$

where, by Th. 1.3 from Chapter III and by Def. 1.1 from I, we conclude that

---

[1] See, also [Rusakov 1992].

$a^{<-1>} = \mathbf{e}(\overset{n-2}{a})$. *Hence, by (3) we have* $a^{<-1>} = a^{\circ}$.

$\overline{4}\ s = -k,\ k > 2:\ \overset{k}{A}(a^{<-k>}, \overset{k(n-1)}{a}) = a \Longleftrightarrow$

$$\overset{k}{A}(a^{<-k>}, \overset{n-2}{a}, \overset{(k-2)(n-1)+1}{a}, \overset{n-2}{a}, a) = a \overset{6VI}{\Longrightarrow}$$

$$\overset{2}{A}(a^{<-k>}, \overset{n-2}{a}, \overset{k-2}{A}(\overset{(k-2)(n-1)+1}{a}), \overset{n-2}{a}, a) = a \overset{(1)(2)}{\Longrightarrow}$$

$$\overset{2}{A}(a^{<-k>}, \overset{n-2}{a}, a^{k-1}, \overset{n-2}{a}, a) = a,$$

where, by Th. 1.3 from Chapter III and by Def. 1.1 from I, we conclude that the following equality holds

$$(\overset{n-2}{a}, a^{k-1})^{-1} = a^{<-k>}.$$

Hence, by (4), we have $a^{<-k>} = a^{-k+1}$.

The proof of c) : By Th. 1.4 and by b).

2 Three more propositions on $n-$groups for $n \geq 3$

2.1. Theorem *[Ušan, Žižović 2002/1]: Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

(1) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

(2) $A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x)$,

(3) $\beta A(x, c_1^{n-2}, \boldsymbol{\alpha}(c_1^{n-2})) = x$ *and*

(4) $\beta A(x_1^n) = A(x_1^{n-1}, \beta(x_n)) = A(x_1^{n-2}, \beta(x_{n-1}), x_n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

Proof. *a)* $\Rightarrow$: Let (Q, A) be an $n-$group and let $\mathbf{e}$ be its $\{1, n\}-$neutral operation ($n \geq 3$). Whence, by Prop. 1.1 from Chapter IV, we conclude that there is $(n-2)-$ary operation $\boldsymbol{\alpha}[= \mathbf{e}]$ and unary operation $\beta[= \{(x, x) | x \in Q\}]$ such that the laws (1)-(4) hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

b) $\Leftarrow$: Let $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ be an algebra of the type $< n, n-2, 1 >$ in which the laws $(1)-(4)$ hold. By the assumption that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws (2) - (4) hold, we conclude that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ also the following laws hold

(5) $A(x, a_1^{n-2}, \beta\boldsymbol{\alpha}(a_1^{n-2})) = x$ and

(6) $A(b_1^{n-2}, \beta\boldsymbol{\alpha}(b_1^{n-2}), x) = x$.

Since the laws (1), (5) and (6) hold in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$, by Prop. 1.1 from Chapter XII, we conclude that (Q, A) is an $n-$group. □

Cf. Chapter X.

Similarly, it is possible to prove that the following proposition holds:

2.2. Theorem *[Ušan, Žižović 2002/1]: Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

$(\overline{1})$ $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

$(\overline{2})$ $A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})$,

$(\overline{3})$ $\beta A(\boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}, x) = x$ *and*

$(\overline{4})$ $\beta A(x_1^n) = A(\beta(x_1), x_2^n) = A(x_1, \beta(x_2), x_3^n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

2.3. Theorem *[Ušan, Žižović 2002/1]: Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

$(\widehat{1})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$ [2]

$(\widehat{2})$ $A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)$,

$(\widehat{3})$ $\beta A(x, c_1^{n-2}, \boldsymbol{\alpha}(c_1^{n-2})) = x$ *and*

$(\widehat{4})$ $\beta A(x_1^n) = A(x_1^{n-1}, \beta(x_n)) = A(\beta(x_1), x_2^n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

Proof. $\widehat{a})$ $\Rightarrow$: Let (Q, A) be an $n-$group and let $\mathbf{e}$ be its $\{1, n\}-$neutral operation ($n \geq 3$). Whence, we conclude that there is $(n-2)-$ary operation $\boldsymbol{\alpha}[= \mathbf{e}]$ and unary operation $\beta[= \{(x, x)|x \in Q\}]$ such that the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws $(\widehat{1})$ – $(\widehat{4})$ hold.

[2]or: $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-2}, A(x_n^{2n-1}))$.

$\widehat{b})$ $\Leftarrow$: Let $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ be an algebra of the type $< n, n-2, 1 >$ in which the laws $(\widehat{1}) - (\widehat{4})$ hold. By the assumption that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws $(\widehat{2}) - (\widehat{4})$ hold, we conclude that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ also the following laws hold

$(\widehat{5})$ $\quad A(x, a_1^{n-2}, \beta\boldsymbol{\alpha}(a_1^{n-2})) = x$ and

$(\widehat{6})$ $\quad A(\beta\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x) = x.$

Since in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws $(\widehat{1})$, $(\widehat{5})$ and $(\widehat{6})$ hold, by Th. 2.2 from Chapter IX, we conclude that (Q, A) is an $n-$group. $\square$

3 On congruence classes of finite $n-$groups for $n \geq 3$

3.1. Theorem *[Ušan 2002/2]: Let (Q, A) be an $n-$group, $|Q| \in N\backslash\{1\}$ and $n \geq 3$. Further on, let θ be an arbitrary congruence of the $n-$group (Q, A) and let $C_t[t \in C_t]$ be an arbitrary class from the set Q/θ. Then there is a $k \in N$ such that the pair $(C_t, \overset{k}{A})$ is a $(k(n-1)+1)-$subgroup of the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$.*[3]

Proof. The following statements hold:

$^\circ 1$ If $(\{Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to the $n-$group (Q, A), then for every $k \in N$ $(\{Q, \{\cdot, \varphi, b^k\})$ is a $(k(n-1)+1)HG-$algebra associated to the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$. *[*Cf. the proof of Th. 7.1 from Chapter VI.*]*

$^\circ 2$ Let (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation and let $n \geq 3$. Also, let θ be an arbitrary element of the set $Con(Q, A)$. Then for every $C_t \in Q/\theta$ there is a sequence c_1^{n-2} over Q such that

(o) $\quad \mathbf{e}(c_1^{n-2}) = t \ (\in C_t).$

*[*Cf. Prop. 1.4 from Chapter IV.*]*

[3]Cf. Remark 2.3 from Chapter XIV.

$^\circ 3$ Let the sequence c_1^{n-2} over Q satisfies (o) from $^\circ 2$. Then the algebra $(\{Q,\{\cdot,\varphi,b\})$ defined with

(1) $x\cdot y \overset{def}{=} A(x,c_1^{n-2},y)$,

(2) $\varphi(x)\overset{def}{=} A(\mathbf{e}(c_1^{n-2}),x,c_1^{n-2})$ and

(3) $b\overset{def}{=} A(\overline{\mathbf{e}(c_1^{n-2})}^{\,n}|)\ [= A(\overset{n}{t})]$

is an $nHG-$algebra associated to the $n-$group (Q,A). *[*Cf. Th. 3.1 from Chapter IV.*]*

$^\circ 4$ $(C_t,\cdot)$ is a subgroup of the group $(Q,\cdot)$. *[*Cf. the proof of Th. 5.5 from Chapter VI.*]*

$^\circ 5$ $\mathbf{e}(c_1^{n-2})$ is a neutral element of the group $(Q,\cdot)$. *[*Cf. Th. 1.3 from Chapter III and by (1).*]*

$^\circ 6$ $(\exists k\in N)b^k = \mathbf{e}(c_1^{n-2})$. *[*By $(o),(1),5^\circ$ and by $|Q|\in N$.*]*

Finally, by $^\circ 1 -^\circ 6$ and by Th. 5.5 $[-(iii)]$ from Chapter VI, we conclude that Th. 3.1 holds. □

4 The $n-$ary case of a Bruck-Hughes Theorem

4.1. Theorem *[Bruck 1946, Hughes 1957]*[4] *: Let $(Q,\cdot)$ be a groupoid with a neutral element e, and let $(Q,\circ)$ be a semigroup. Let also α,β,γ be the permutations of the set Q such that*

(o) $\quad x\circ y = \gamma(\alpha(x)\cdot\beta(y))$ *for all* $x,y\in Q$.

Then $(Q,\circ)$ is a semigroup with a neutral element[5].

Proof.[6] Since $(Q,\circ)$ is a semigroup, by (o), we have

---

[4] [Kurosh 1962].

[5] Precisely: Then $(Q,\circ)$ is isomorphic to $(Q,\cdot)$, whence $(Q,\circ)$ and $(Q,\cdot)$ are semigroup with neutral elements.

[6] [Hughes 1957].

$\gamma(\alpha(\gamma(\alpha a \cdot \beta b)) \cdot \beta c) = \gamma(\alpha a \cdot \beta(\gamma(\alpha b \cdot \beta c)))$,

whence, since γ is a permutation in the set Q, we obtain

(1) $\alpha(\gamma(\alpha a \cdot \beta b)) \cdot \beta c) = \alpha a \cdot \beta(\gamma(\alpha b \cdot \beta c))$.

Putting $\alpha a = \beta c = e$ in (1), we obtain

(2) $\alpha(\gamma(\beta b)) = \beta(\gamma(\alpha b))$ for all $b \in Q$.

In addition, putting $\alpha a = e$ in (1) and using (2), we have

$\beta(\gamma(\alpha b)) \cdot \beta c = \beta(\gamma(\alpha b \cdot \beta c))$,

whence, putting $\alpha b = a$ and $\beta c = b$, we obtain

(3) $\beta(\gamma a) \cdot b = \beta(\gamma(a \cdot b))$ for all $a, b \in Q$.

Similarly, putting $\beta c = e$ in (1) and using (2), we have

$\alpha(\gamma(\alpha a \cdot \beta b)) = \alpha a \cdot \alpha(\gamma(\beta b))$,

whence, letting a instead αa and b instead βb, we obtain

(4) $\alpha(\gamma(a \cdot b)) = a \cdot \alpha(\gamma b)$ for all $a, b \in Q$.

Finally, by $(o), (2), (3)$ and (4), we have

$$\begin{aligned} \alpha(\gamma(\beta(a \circ b))) &\overset{(0)}{=} \alpha(\gamma(\beta(\gamma(\alpha a \cdot \beta b)))) \\ &\overset{(3)}{=} \alpha(\gamma(\beta(\gamma(\alpha a)) \cdot \beta b)) \\ &\overset{(4)}{=} \beta(\gamma(\alpha a)) \cdot \alpha(\gamma(\beta b)) \\ &\overset{(2)}{=} \alpha(\gamma(\beta a)) \cdot \alpha(\gamma(\beta b)) \end{aligned}$$

for all $a, b \in Q$. $\square$

4.2. Theorem *[Ušan 1999/5]: Let (Q, A) be an $n-$groupoid, let (Q, B) be an $n-$groupoid, let α, β, γ be the $(n-1)-$ary operations in the set Q, and let $n \geq 3$. Moreover, let the following statements hold:*

(1_1) *(Q, A) is an $<1, n>-$associative $n-$groupoid;*

(1_2) *(Q, A) is an $<1, 2>-$associative [or $<n-1, n>-$associative] $n-$groupoid;*

(2) *(Q, B) has a $\{1, n\}-$neutral operation* **e**;

(3) *For every sequence a_1^{n-1} over Q there are exactly one x, exactly one y and exactly one z such that*

$\alpha(a_1^{n-2}, x) = a_{n-1}$, $\beta(a_1^{n-2}, y) = a_{n-1}$ *and* $\gamma(a_1^{n-2}, z) = a_{n-1}$; *and*

(4) *For all $x, y \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds*

$$A(x, a_1^{n-2}, y) = \gamma(a_1^{n-2}, B(\alpha(a_1^{n-2}, x), a_1^{n-2}, \beta(a_1^{n-2}, y))).$$

Then (Q, A) is an $n-$group[7]

Proof. We prove that the following five assertions hold:

1° Let a_1^{n-2} be an arbitrary sequence over Q and let for all $x, y \in Q$

(a) $x \cdot y = B(x, a_1^{n-2}, y)$ and

(b) $x \circ y = A(x, a_1^{n-2}, y)$.

Then $(Q, \cdot)$ is a groupoid with the neutral element $e = \mathbf{e}(a_1^{n-2})$, and $(Q, \circ)$ is a semigroup.

2° Let a_1^{n-2} be a sequence over Q from 1°. Also let for all $z \in Q$

(c) $\quad \widehat{\alpha}(z) \overset{def}{=} \alpha(a_1^{n-2}, z),\ \widehat{\beta}(z) \overset{def}{=} \beta(a_1^{n-2}, z)$ and $\widehat{\gamma}(z) \overset{def}{=} \gamma(a_1^{n-2}, z)$.

Then: $a)$ for every $x, y \in Q$ the following equality holds

(d) $x \circ y = \widehat{\gamma}(\widehat{\alpha}(x) \cdot \widehat{\beta}(y))$, and

$b)$ $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ are permutations in the set Q.

3° Let a_1^{n-2} sequence over Q from 1°(2°). Also let for all $x \in Q$

(e) $F(a_1^{n-2}, x) \overset{def}{=} \alpha(a_1^{n-2}, \gamma(a_1^{n-2}, \beta(a_1^{n-2}, x)))$.

Then for all $x, y \in Q$ the following equality holds

(f) $F(a_1^{n-2}, A(x, a_1^{n-2}, y)) = B(F(a_1^{n-2}, x), a_1^{n-2}, F(a_1^{n-2}, y))$.

4° Let a_1^{n-2} be an arbitrary sequence over Q, let $\mathbf{e}$ be an $\{1, n\}-$neutral operation in the $n-$groupoid (Q, B) *[:(2)]*, and let F be from 3° *[:(e)]*. Then, the equation

(g) $\quad F(a_1^{n-2}, z) = b$

with the unknown z has exactly one solution in Q.

5° Let E be a mapping of the set Q^{n-2} into the set Q such that for every sequence a_1^{n-2} over Q the following equality holds

(h) $\quad F(a_1^{n-2}, \mathsf{E}(a_1^{n-2})) = \mathbf{e}(a_1^{n-2})$.

[7] i.e., (Q, A) is an $n-$semigroup with an $\{1, n\}-$neutral operation (:Th. 2.1 from Chapter IX.). Cf. Th. 4.1.

Then E is an $\{1,n\}-$neutral operation of the $n-$groupoid (Q,A).

Sketch of the proof of 1° :

$$1)\quad x\cdot \mathbf{e}(a_1^{n-2}) \overset{(a)}{=} B(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})) \overset{(2)}{=} x,$$
$$\mathbf{e}(a_1^{n-2})\cdot x \overset{(a)}{=} B(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x) \overset{(2)}{=} x.$$

$$\begin{aligned} 2)\quad (x\circ y)\circ z &\overset{(b)}{=} A(A(x,a_1^{n-2},y),a_1^{n-2},z)\\ &\overset{(1_1)}{=} A(x,a_1^{n-2},A(y,a_1^{n-2},z))\\ &\overset{(b)}{=} x\circ(y\circ z)\end{aligned}$$

Sketch of the proof of 2° :

$$\begin{aligned} \overline{a})\quad (x\circ y) &\overset{(b)}{=} A(x,a_1^{n-2},y)\\ &\overset{(4)}{=} \gamma(a_1^{n-2},B(\alpha(a_1^{n-2},x),a_1^{n-2},\beta(a_1^{n-2},y)))\\ &\overset{(a)(c)}{=} \widehat{\gamma}(\widehat{\alpha}(x)\cdot\widehat{\beta}(y)).\end{aligned}$$

$\overline{b}$) By (3) and by (c).

Sketch of the proof of 3° :

By $1^\circ, 2^\circ$ and by the proof of Theorem 4.1.

Sketch of the proof of 4° :

By (2), by 3°-(e) and by (3).

Sketch of the proof of 5° :

$$F(a_1^{n-2},A(x,a_1^{n-2},\mathsf{E}(a_1^{n-2})))\overset{3^\circ(f)}{=}$$
$$B(F(a_1^{n-2},x),a_1^{n-2},F(a_1^{n-2},\mathsf{E}(a_1^{n-2})))\overset{(h)}{=}$$
$$B(F(a_1^{n-2},x),a_1^{n-2},\mathbf{e}(a_1^{n-2}))\overset{(2)}{=}F(a_1^{n-2},x),$$

where, by 4°, we have

$$A(x,a_1^{n-2},\mathsf{E}(a_1^{n-2}))=x.$$

Similarly, we have

$$A(\mathsf{E}(a_1^{n-2}),a_1^{n-2},x)=x.$$

Finally, by (1_2), by 5° and by Th. 2.2 from Chapter IX, we conclude that (Q,A) is an $n-$group. $\square$

4.3. Reamark: *Belousov V. D. and Sandik M. D. have proved the following assertion [Belousov, Sandik 1966]: Let (Q, B) be an $n-$loop, $n \geq 3$ and let (Q, A) be an $n-$group with (at least one) neutral element [cf. Chapter II-1]. Also let $\alpha_1, \ldots, \alpha_{n+1}$ be the permutations in the set Q such that for every $x_1^n \in Q$ the following equality holds*

$$A(x_1^n) = \alpha_{n+1}B(\alpha_1(x_1), \ldots, \alpha_n(x_n)).$$

Then (Q, A) is isomorphic to (Q, B).

5 On $(m, n)-$rings

5.1. Definition *[Boccioni 1965, Čupona 1965, Crombez 1972/1]: Let (Q, A) be a commutative $m-$group and $m \geq 2$. Let also (Q, M) be an $n-$groupoid and $n \geq 2$. We say that (Q, A, M) is an $(m, n)-$**ring** iff for all $i \in \{1, \ldots, n\}$ and for every $a_1^{n-1}, b_1^m \in Q$ the following equality holds*

(0) $M(a_1^{i-1}, A(b_1^m), a_i^{n-1}) = A(\overline{\,M(a_1^{i-1}, b_j, a_i^{n-1})\,}\big|_{j=1}^{m})$[8].

5.2. Theorem *[Ušan, Žižović 1999/2]: Let (Q, A, M) be an $(m, n)-$ring and let $\mathbf{O}$ the $\{1, m\}-$neutral operation of the $m-$group (Q, A). Then for all $i \in \{1, \ldots, n\}$, for every $a_1^{n-1} \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds*

(1) $M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) = \mathbf{O}(\overline{\,M(a_1^{i-1}, c_j, a_i^{n-1})\,}\big|_{j=1}^{m-2})$[9]

Proof. Let

(2) $A^{-1}(x, c_1^{m-2}, y) = z \overset{def}{\Longleftrightarrow} A(x, c_1^{m-2}, z) = y$

for all $x, y, z \in Q$ and for every sequence c_1^{m-2} over Q. Then the following statements hold:

1° For all $i \in \{1, \ldots, n\}$, for every $a_1^{n-1}, x, y \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds

[8] Cf. Appendix I-2.

[9] $\overline{\,M(a_1^{i-1}, c_j, a_i^{n-1})\,}\big|_{j=1}^{0} \overset{def}{=} \emptyset.$

$$M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, y), a_i^{n-1})$$
$$= A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1}))$$

2° For all $x \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds

$$A^{-1}(x, c_1^{m-2}, x) = \mathbf{O}(c_1^{m-2}).$$

Sketch of the proof of 1° :

$$A(x, c_1^{m-2}, z) = y \Rightarrow$$
$$M(a_1^{i-1}, A(x, c_1^{m-2}, z), a_i^{n-1}) = M(a_1^{i-1}, y, a_i^{n-1}) \overset{(0)}{\Longleftrightarrow}$$
$$A(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, z, a_i^{n-1}))$$
$$= M(a_1^{i-1}, y, a_i^{n-1}) \overset{(2)}{\Longleftrightarrow}$$
$$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{(M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1}))$$
$$= M(a_1^{i-1}, z, a_i^{n-1})) \overset{(2)}{\Longleftrightarrow}$$
$$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{(M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1}))$$
$$= M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, y), a_i^{n-1}).$$

Sketch of the proof of 2° :

$$A(x, c_1^{m-2}, \mathbf{O}(c_1^{m-2}))^{1.3III}\!\!= x \overset{(2)}{\Longleftrightarrow}$$
$$A^{-1}(x, c_1^{m-2}, x) = \mathbf{O}(c_1^{m-2}).$$

Finally, by 1° and by 2°, we conclude that for all $i \in \{1, \ldots, n\}$, for every a_1^{n-1}, $x \in Q$ and for every sequence c_1^{m-2} over Q the following series of equalities holds:

$$M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) \overset{2^\circ}{=} M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, x), a_i^{n-1})$$
$$\overset{1^\circ}{=} A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, x, a_i^{n-1}))$$
$$\overset{2^\circ}{=} \mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}).$$

Remark: For $m = n = 2 : a \cdot \mathbf{O}(\emptyset) = \mathbf{O}(\emptyset) \cdot a = \mathbf{O}(\emptyset)$. □

5.3. Remark: $\mathbf{O}$ *is an* $\{i,j\}-$*neutral operation of the* $m-$*group* (Q,A) *for every* $\{i,j\} \subseteq \{1,\ldots,m\}$, $i<j$. *Cf. Def. 5.1 and Chapter V.*

5.4. Theorem *[Ušan, Žižović 1999/2]: Let* (Q,A,M) *be an* $(m,n)-$*ring and let* $-$ *be the inverse operation in the* $m-$*group* (Q,A). *Then for all* $i \in \{1,\ldots,n\}$, *for every* $a_1^{n-1}, b \in Q$ *and for every sequence* c_1^{m-2} *over* Q *the following equality holds*

$$M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1}) = -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, b, a_i^{n-1})).$$

Proof. Firstly we prove that under the assumption the following statements hold:

$^\circ 1$ For all $i \in \{1,\ldots,n\}$, for every $a_1^{n-1}, b \in Q$ and for every sequence c_1^{m-2} over Q we have

$$\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}) =$$
$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1})).$$

$^\circ 2$ For all $i \in \{1,\ldots,n\}$, for every $a_1^{n-1}, b \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds

$$\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}) =$$
$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2},$$
$$M(a_1^{i-1}, b, a_i^{n-1}))).$$

Sketch of the proof of $^\circ 1$:

$$\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}) \overset{5.2}{=} M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) \overset{1.3III}{=}$$
$$M(a_1^{i-1}, A(b, c_1^{m-2}, -(c_1^{m-2}, b)), a_i^{n-1}) \overset{5.1}{=}$$
$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1})).$$

Sketch of the proof of $^\circ 2$:

$$\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}) \overset{1.3III}{=}$$
$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2},$$
$$M(a_1^{i-1}, b, a_i^{n-1}))).$$

Finally, by °1 and by °2, we conclude that for all $i \in \{1, \ldots, n\}$, for every $a_1^{n-1}, b \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds

$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1})) =$$
$$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2},$$
$$M(a_1^{i-1}, b, a_i^{n-1}))),$$

whence, by Def. 1.1 from Chapter I, we conclude that for all $i \in \{1, \ldots, n\}$, for every $a_1^{n-1}, b \in Q$ and for every sequence c_1^{m-2} over Q the following equality holds

$$M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1}) = -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\,\big|_{j=1}^{m-2}, M(a_1^{i-1}, b, a_i^{n-1})).$$

This completes the proof.
Remark: For $m = n = 2$: $a \cdot (-b) = -(a \cdot b)$. □

5.5. Remark: *About the $(m, n)-$rings see also, for example, in: [Iancu 1999] and [Paunic 1985].*

Chapter XVI

ON $(n, m)-$GROUPS, NP-POLYAGROUPS AND POLYAGROUPS

1 On $(n, m)-$groups

1.1. Definition *[Čupona 1983]: Let* (Q, A) *be an* $(n, m)-$*groupoid* $(A : Q^n \longrightarrow Q^m)$ *and let* $n \geq m + 1$ $(n, m \in N)$. *Then:* (a) *we say that* (Q, A) *is an* $(n, m)-$**semigroup** *iff for every* $i, j \in \{1, \ldots, n - m + 1\},\ i < j$, *the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$

[:$< i, j > -$associative law]; and (b) *we say that* (Q, A) *is an* $(n, m)-$**group** *iff* (Q, A) *is an* $(n, m)-$*semigroup and for every* $a_1^n \in Q$ *there is exactly one sequence* x_1^m *over* Q *and exactly one sequence* y_1^m *over* Q *such that the following equalities hold*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

1.2. Remark: *A notion of an* $(n, m)-$*group was introduced by Ǵ. Čupona in [Čupona 1983] as a generalization of the notion of a group (*$n-$*group). The paper [Čupona, Celakoski, Markovski, Dimovski 1988] is mainly a survey on the known results for vector valued groupoids, semigroups and groups (up to 1988).*

1.3. Definition *[Ušan 1989]: Let* (Q, A) *be an* $(n, m)-$*groupoid and* $n \geq 2m$. *Let also* **e** *be a mapping of the set* Q^{n-2m} *into the set* Q^m. *Then, we say that* **e** *is an* $\{1, n - m + 1\}-$**neutral operation** *of the* $(n, m)-$*groupoid* (Q, A) *iff for all* $x_1^m \in Q^m$ *and for every sequence* a_1^{n-2m} *over* Q *the following equalities hold*

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m \text{ and } A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

Remark: For $m = 1$, $\mathbf{e}$ is an $\{1, n\}-$neutral operation of the $n-$groupoid (Q, A). Cf. Chapter II-2.

1.4. Proposition *[Ušan 1989]: Every $(n,m)-$groupoid $(n \geq 2m)$ has at most one $\{1, n-m+1\}-$neutral operation ($\{i,j\}-$neutral operation).*
Cf. the proof of Prop. 2.3 from Chapter II.

1.5. Theorem *[Ušan 1999/3]: Let $n \geq 2m, m \geq 2$ and let (Q, A) be an $(n,m)-$groupoid. Then, (Q, A) is an $(n,m)-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

$$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}),$$

$$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$$

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m \text{ and}$$

$$A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m})$$

hold in the algebra $Q, \{A,^{-1}, \mathbf{e}\})$.
See the proof in [Ušan 2005/1].

1.6. Theorem *[Ušan 1999/3]: Let $n \geq 3m$ and let (Q, A) be an $(n,m)-$groupoid. Then, (Q, A) is an $(n,m)-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(a) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}),$

(b) $A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m}),$

(c) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$ *and*

(d) $A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m})$

hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$.
See the proof in [Ušan 2005/1].

Remark: For $m = 1$: $(a) = (b)$.

1.7. Theorem *[Ušan 1999/3]: Let $n \geq 2m$ and let (Q,A) be an $(n,m)-$groupoid. Then, (Q,A) is an $(n,m)-$group iff there are mappings $^{-1}$ and* $\mathbf{e}$, *respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

$$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}),$$

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m \text{ and}$$

$$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$$

hold in the algebra $Q, \{A, ^{-1}, \mathbf{e}\})$.
See the proof in [Ušan 2005/1].

1.8. Theorem *[Ušan 1999/3]: Let $n \geq 2m$ and let (Q,A) be an $(n,m)-$groupoid. Then, (Q,A) is an $(n,m)-$group iff there is a mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the laws*

$$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})\ ^1,$$

$$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m \text{ and}$$

$$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$$

hold in the algebra $Q, \{A, ^{-1}\})$.
See the proof in [Ušan 2005/1].

The following two propositions also hold.

1.9. Theorem *[Ušan 2000]: Let $n \geq 2m$ and let (Q,A) be an $(n,m)-$groupoid. Then, (Q,A) is an $(n,m)-$group iff the following statements hold:*

(i) (Q,A) is an $< 1, n-m+1 > -$associative $(n,m)-$groupoid,

(ii) (Q,A) is an $< 1, 2 > -$associative $(n,m)-$groupoid [2]*,*

(iii) For every $a_1^n \in Q$ there is **at least one** $x_1^m \in Q^m$ *and* **at least one** $y_1^m \in Q^m$ *such that the following equalities hold*

[1]or: $A(x_1^{n-m-1}, A(x_{n-m}^{2n-m+1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$.
[2]or: $< n-m, n-m+1 > -$associative $(n,m)-$groupoid.

$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ *and* $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$.
See the proof in [Ušan 2005/1].

1.10. Theorem *[Ušan 2001/2]: Let* $n \geq 3m$ *and let* (Q, A) *be an* $(n, m)-$*groupoid. Then,* (Q, A) *is an* $(n, m)-$*group iff there is* $i \in \{m + 1, \ldots, n - 2m + 1\}$ *such that the following statements hold*
(a) *The* $< i - 1, i > -$*associative law holds in* (Q, A),
(b) *The* $< i, i + 1 > -$*associative law holds in* (Q, A) *and*
(c) *For every* $a_1^n \in Q$ *there is* **exactly one** $x_1^m \in Q^m$ *such that*

$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n$.
See the proof in [Ušan 2005/1].

2 On NP-polyagroups and polyagroups

2.1. Definition *[Ušan, Galić 2001]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then: we say that* (Q, A) *is an* $NP-$**polyagroup of the type** $(s, n - 1)$ *iff the following statements hold:*

1° *For all* $i, j \in \{1, \ldots, n\}$ $(i < j)$ *if* $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$, *then the* $< i, j > -$*associative law holds in* (Q, A); *and*

2° *For all* $i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ *and for every* $a_1^n \in Q$ *there is exactly one* $x_i \in Q$ *such that the equality*

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds.[3]

2.2. Theorem *[Ušan, Galić 2001]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then:* (Q, A) *is an* $NP-$**polyagroup of the type** $(s, n - 1)$ *iff there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra*

[3] For $s = 1$ (Q, A) is a $(k + 1)-$group, where $k + 1 \geq 3;\ k > 1$.

$(Q, \{A,^{-1}, \mathbf{e}\})$ *[of the type $< n, n-1, n-2 >$]:*

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1}),$$

$$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x \text{ and}$$

$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}).$$

See the proof in [Ušan 2005/2].
Cf. Th. 3.4(-Tabl. 2) from Chapter III.

2.3. Theorem *[Ušan, Galić 2001]: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an $NP-$***polyagroup of the type** *$(s, n-1)$ iff there are mappings $^{-1}$ and* **e**, *respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $Q, \{A,^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]:*

$$A(x_1^{(k-1)\cdot s}, A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1})),$$

$$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \text{ and}$$

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}).$$

Cf. Th. 3.1 from Chapter III.

2.4. Theorem *[Ušan, Galić 2001]: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an $NP-$***polyagroup of the type** *$(s, n-1)$ iff there are mappings $^{-1}$ and* **e**, *respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $Q, \{A,^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]:*

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})^4,$$

$$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x \text{ and}$$

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x.$$

See the proof in [Ušan 2005/2].
Cf. Th. 3.4(-Tabl. 6) from Chapter III.

The following two operations also hold.

---

[4]or: $A(x_1^{(k-1)\cdot s}, A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1}))$.

2.5. Theorem *[Ušan, Žižović 2001]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then,* (Q, A) *is an* $NP-$**polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

(i) (Q, A) *is an* $< 1, n > -$*associative* $n-$*groupoid,*

(ii) (Q, A) *is an* $< 1, s+1 > -$*associative* $n-$*groupoid* **or** (Q, A) *is a* $< (k-1) \cdot s + 1, k \cdot s + 1 > -$*associative* $n-$*groupoid, and*

(iii) *For every* $a_1^n \in Q$ *there are* **at least one** $x \in Q$ *and* **at least one** $y \in Q$ *such that the following equalities hold*

$A(a_1^{n-1}, x) = a_n$ *and* $A(y, a_1^{n-1}) = a_n$.

See the proof in [Ušan 2005/2].

Cf. Chapter IX-3.

2.6. Theorem *[Ušan 2002/1]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then,* (Q, A) *is an* $NP-$**polyagroup of the type** $(s, n-1)$ *iff there is* $i \in \{t \cdot s + 1 | t \in \{1, \ldots, k-1\}$ *such the following statements hold:*

(a) *The* $< i-s, i > -$*associative law holds in* (Q, A),

(b) *The* $< i, i+s > -$*associative law holds in* (Q, A) *and*

(c) *For every* $a_1^n \in Q$ *there is* **exactly one** $x \in Q$ *such that* $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

See the proof in [Ušan 2005/2].

Cf. Chapter IX-3.

2.7. Definition *[Sokhatski 1998; Sokhatsky, Yurevich 1999]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then, we say that* (Q, A) *is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

$^\circ 1$ *For all* $i, j \in \{1, \ldots, n\}(i < j)$ *if* $i \equiv j (mod\, s)$*, then the* $< i, j > -$*associative law holds in* (Q, A)*; and*

$^\circ 2$ (Q, A) *is an* $n-$*quasigroup.*

2.8. Proposition: *Every polyagroup of the type* $(s, n-1)$ *is an* $NP-$*polyagroup* *of the type* $(s, n-1)$. *[By Def. 2.1 and by Def. 2.7.]*

2.9. Theorem *[Ušan, Žižović 2002/2]: Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then,* (Q, A) *is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

(i) (Q, A) *is an* $< i, s+i > -$*associative* $n-$*groupoid for all* $i \in \{1, \ldots, s\}$*;*

(ii) (Q, A) *is an* $< 1, n > -$*associative* $n-$*groupoid;*

(iii) For every $a_1^n \in Q$ *there are* **at least one** $x \in Q$ *and* **at least one** $y \in Q$ *such that the following equalities hold*

$$A(x, a_1^{n-1}) = a_n \text{ and } A(a_1^{n-1}, y) = a_n; \text{ and}$$

(iv) For every $a_1^n \in Q$ *and for all* $j \in \{2, \ldots, s\} \cup \{(k-1) \cdot s + 2, \ldots, k \cdot s\}$ *there is* **exactly one** $x_j \in Q$ *such that the following equality holds*

$$A(a_1^{j-1}, x_j, a_j^{n-1}) = a_n.$$

See the proof in [Ušan 2005/2].
See, also Th. 3.2 –IX.

Appendix 1

About the expression a_p^q

Let $p \in N,\ q \in N \cup \{0\}$ and let a be a mapping of the set $\{i | i \in N,\ i \geq p\ \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q & ; p < q \\ a_p & ; p = q \\ \textit{empty sequence}(= \emptyset) & ; p > q. \end{cases}$$

For example: $X(a_1^{j-1}, Y(a_j^{j+n-1}), a_{j+n}^{2n-1}),\ j \in \{1, \ldots, n\},\ n \geq 3$, for $j = n$ stands for $X(a_1, \ldots, a_{n-1}, Y(a_n, \ldots, a_{2n-1}))$.
Besides, in some situations **instead of** a_p^q **we write** $(a_i)_{i=p}^q$ [briefly: $(a_i)_p^q$]. For example: $(\forall x_i \in Q)_1^q$ for $q > 1$ stands for $\forall x_1 \in Q \ldots \forall x_q \in Q$ [usualy, we write: $(\forall x_1 \in Q) \ldots (\forall x_q \in Q)$], for $q = 1$ it stands for $\forall x_1 \in Q$, and for $q = 0$ it stands for an empty sequence $(= \emptyset)$. If a_p^q is a sequence over a set $S, p \leq q$ and the equalities $a_p = \ldots = a_q = b\ (\in S)$ are satisfied, then

$$a_p^q \text{ is denoted by } \overset{q-p+1}{b}.$$

In connection with this, if $q - p + 1 = r$ (when we assume that there would be no missunderstanding),

$$\text{instead of } \overset{q-p+1}{b} \text{ we write } \overset{r}{b}.$$

In addition, we denote the empty sequence over S with $\overset{\circ}{b}$, where b is an arbitrary element from S.

Appendix 2

On skew operation of an $n-$group

1. Definition *[Dörnte 1928]: Let (Q, A) be an $n-$group and $n \geq 3$. Then: we say that an 1-ary operation* $^{-}$ *is a* **skew operation of the $n-$group** *(Q, A) iff for each $a \in Q$ following equality holds*

(o) $\quad A(\overset{n-1}{a}, \overline{a}) = a.$[1]

2. Proposition *[Dörnte 1928]: Let (Q, A) be an $n-$group and $n \geq 3$. Then for all $i \in \{1, \ldots, n\}$ and for every $a \in Q$ the following equality holds*

$$A(\overset{i-1}{a}, \overline{a}, \overset{n-i}{a}) = a.$$

Sketch of the proof.

$$A(\overset{n-1}{a}, \overline{a}) \overset{(o)}{=} a \Rightarrow$$

$$A(\overset{i-1}{a}, A(\overset{n-1}{a}, \overline{a}), \overset{n-i}{a}) = A(\overset{n}{a}) \overset{1.1I}{\Longrightarrow}$$

$$A(\overset{i-1}{a}, \overset{n-i}{a}, A(\overset{i-1}{a}, \overline{a}, \overset{n-i}{a})) = A(\overset{n}{a}) \Longrightarrow$$

$$A(\overset{n-1}{a}, A(\overset{i-1}{a}, \overline{a}, \overset{n-i}{a})) = A(\overset{n-1}{a}, a) \overset{1.1I}{\Longrightarrow}$$

$$A(\overset{i-1}{a}, \overline{a}, \overset{n-i}{a}) = a. \quad \square$$

3. Proposition *[Dörnte 1928]: Let (Q, A) be an $n-$group and $n \geq 3$. Then for all $a, x \in Q$ the equality*

$$A(x, \overset{n-2}{a}, \overline{a}) = x$$

holds.

Sketch of the proof.

$$A(x, \overset{n-2}{a}, \overline{a}) = y \Rightarrow$$

$$A(A(x, \overset{n-2}{a}, \overline{a}), \overset{n-1}{a}) = A(y, \overset{n-1}{a}) \Rightarrow$$

$$A(x, \overset{n-2}{a}, A(\overline{a}, \overset{n-1}{a})) = A(y, \overset{n-1}{a}) \overset{2}{\Rightarrow}$$

$$A(x, \overset{n-2}{a}, a) = A(y, \overset{n-1}{a}) \overset{1.1I}{\Longrightarrow} x = y. \quad \square$$

---

[1]or: $\overline{a} = A^{-1}(\overset{n}{a})$, where $A^{-1}(x_1^{n-1}, y) = z \overset{def}{\Longleftrightarrow} A(x_1^{n-1}, z) = y$.

4. Proposition: *Let $n \geq 3$, (Q, A) be an $n-$group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation, and* $^{-}$ *its skew operation. Then for all* $a \in Q$ *the following equality holds*

$$\overline{a} = \mathbf{e}(\overset{n-2}{a}).$$

Sketch of the proof.

$$A(x, \overset{n-2}{a}, \overline{a}) \overset{3.}{=} x \text{ and } A(x, \overset{n-2}{a}, \mathbf{e}(\overset{n-2}{a})) \overset{1.3III}{=} x \Rightarrow$$

$$A(x, \overset{n-2}{a}, \overline{a}) \;=\; A(x, \overset{n-2}{a}, \mathbf{e}(\overset{n-2}{a})) \overset{1.1I}{\Longrightarrow} \overline{a} \;= \mathbf{e}(\overset{n-2}{a}). \quad \square$$

The following proposition also holds.

5. Theorem *[Žižović 1998]: Let $n \geq 3$, let (Q, A) be an $n-$group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* $^{-}$ *its skew operation. Then for every sequence* a_1^{n-2} *over* Q *the following equality holds*

$$\mathbf{e}(a_1^{n-2}) = \overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a}_1).$$

*[*Cf. Th. 2.9 from Chapter VIII. For $k = 0$ $\overset{k}{A}(x_1^{k(n-1)+1}) \overset{def}{=} x_1$.*]*

$n-$groups ($n \geq 3$) as algebras of the type $< n, 1 >$ with laws were described in [Gleichgewicht, Glazek 1967]:

6. Theorem *[Gleichgewicht, Glazek 1967]: For $n \geq 3$ an $n-$semigroup (Q, A) is an $n-$group iff there is a unary operation* $^{-}$ *in Q such that the following laws are satisfied:*

$$A(x, \overset{n-2}{a}, \overline{a}) = x, \; A(\overline{a}, \overset{n-2}{a}, x) = x,$$

$$A(x, \overset{n-3}{a}, \overline{a}, a) = x, \; A(a, \overline{a}, \overset{n-3}{a}, x) = x.$$

$n-$groups ($n \geq 3$) as algebras of the type $< n, 1 >$ with laws have been described also, for example, in [Celakoski 1977], [Dudek, Glazek, Gleichgewicht 1977] and [Dudek 1995]. For example:

7. Theorem *[Celakoski 1977]: For $n \geq 3$ an $n-$semigroup (Q, A) is an $n-$group iff for every $a \in Q$ there is a unary operation* $^{-}$ *in Q such that for some $p : 0 \leq p \leq n-2$ and for some: $s : 0 \leq s \leq n-2$ the following* *identities hold:*

$$A(\overset{p}{a},\overline{a},\overset{n-p-2}{a},x) = x \text{ and } A(x,\overset{n-s-2}{a},\overline{a},\overset{s}{a}) = x.$$

[See, also [Čupona, Celakoski 1980].]

8. Theorem *(Hosszú-Gluskin Theorem): Let* $n \geq 3$, (Q,A) *be an* $n-$*group and* $^-$ *its skew operation. Let also* c *be an arbitrary element from the set* Q*, and let*

$$x \cdot y \overset{def}{=} A(x,\overset{n-2}{c},y),$$

$$\varphi(x) \overset{def}{=} A(\overline{c},x,\overset{n-2}{c}) \text{ and}$$

$$b \overset{def}{=} A(\overline{c},\ldots,\overline{c})$$

for all $x,y \in Q$*. Then, the following statements hold:*
(i) $(Q,\{\cdot,\varphi,b\})$ *is an* $nHG-$*algebra, and*
(ii) *For every* $x_1^n \in Q$ *the equality*

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$$

holds. [2]

9. Remark: *Some old unsolved problems connected with skew elements in* $n-$*ary groups are discussed in [Dudek 2001].*

[2] [Sokolov 1976].

Appendix 3

About embedding of $n-$groups into groups

1. Proposition *[Post 1940]: Let $n \geq 3$ and let $(Q;A)$ be an $n-$group. Also, let Γ be the set of all sequences over Q of finite length and let the multiplication $*$ in Γ be defined as the juxtaposition:*

$$(1) \qquad a_1^i \, * \, b_1^j \overset{def}{=} a_1^i, b_1^j$$

for all $a_1^i, b_1^j \in \Gamma;\ i,j \in N \cup \{0\}$. Further on, let θ be a relation in Γ defined as the following equivalence:

$$(2) \quad \boldsymbol{\alpha}\ \theta\ \boldsymbol{\beta} \overset{def}{\Longleftrightarrow} (\exists \boldsymbol{\gamma} \in \Gamma)(\exists \boldsymbol{\delta} \in \Gamma)(\exists k \in N)(\exists l \in N)\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta})$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma$, where $|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| = k(n-1)+1$ and $|\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = l(n-1)+1$. Then the following statements holds:

(i) $(\Gamma;*)$ is a (well-known) semigroup with neutral element $\emptyset$ (empty sequence).

(ii) $\emptyset\ \theta\ \emptyset$.

(iii) For all $\boldsymbol{\alpha} \in \Gamma \backslash \{\emptyset\}$ the following equivalence holds

$\boldsymbol{\alpha}\ \theta\ \emptyset \Leftrightarrow (\exists k \in N)(\exists y \in Q)\overset{k}{A}(\boldsymbol{\alpha}, y) = y.$

[$\emptyset\ \theta\ \boldsymbol{\alpha} \Leftrightarrow (\exists k \in N)(\exists y \in Q)\overset{k}{A}(y, \boldsymbol{\alpha}) = y$].

(iv) θ is a congruence relation in $(\Gamma;*)$.

(v) For every $\boldsymbol{\alpha} \in \Gamma$ there is sequence $\boldsymbol{\beta} \in \Gamma$ such that the following formula holds

$$\boldsymbol{\alpha} * \boldsymbol{\beta}\ \theta\ \emptyset.$$

Sketch of the prof. Firstly we prove the following statements:

$\overset{\circ}{1}$ Let

$\overset{k}{A}(c_1^i, \boldsymbol{\alpha}, d_1^j) = \overset{l}{A}(c_1^i, \boldsymbol{\beta}, d_1^j).$

Then

$$\overset{k+\bar{i}+\bar{j}}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\alpha}, d_1^j, b_1^{\bar{j}(n-1)}) = \overset{l+\bar{i}+\bar{j}}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\beta}, d_1^j, b_1^{\bar{j}(n-1)}),$$

where $a_1^{\bar{i}(n-1)}$ and $b_1^{\bar{j}(n-1)}$ are arbitrary sequences over Q.

$\overset{\circ}{2}$ Let

$$\overset{\bar{k}}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\alpha}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}) = \overset{\bar{l}}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\beta}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}),$$

where $1 \le i \le n-1$ and $1 \le j \le n-1$. Then there are $c \in Q$ and $d \in Q$ such that the following equality holds

$$\overset{\bar{k}-\bar{i}-\bar{j}}{A}(c, c_1^{i-1}, \boldsymbol{\alpha}, d, d_1^{j-1}) = \overset{\bar{l}-\bar{i}-\bar{j}}{A}(c, c_1^{i-1}, \boldsymbol{\beta}, d, d_1^{j-1}).$$

$\overset{\circ}{3}$ Let

$$\overset{k}{A}(c_1^i, a_1^t, d_1^j) = \overset{l}{A}(c_1^i, b_1^s, d_1^j);$$

$i, j, t, s \in N$. Then

$$\overset{k}{A}(\widehat{c}_1^{i+1}, a_1^t, \widehat{d}_1^{j-1}) = \overset{l}{A}(\widehat{c}_1^{i+1}, b_1^s, \widehat{d}_1^{j-1}) \text{ and}$$

$$\overset{k}{A}(\widehat{c}_1^{i-1}, a_1^t, \widehat{d}_1^{j+1}) = \overset{l}{A}(\widehat{c}_1^{i-1}, b_1^s, \widehat{d}_1^{j+1})$$

$\overset{\circ}{4}$ Let

$$\overset{k}{A}(c_1^i, a_1^t, d_1^j) = \overset{l}{A}(c_1^i, b_1^s, d_1^j);$$

$i, j, t, s \in N$. Then

$$\overset{k}{A}(\overline{c}_1^i, a_1^t, \overline{d}_1^j) = \overset{l}{A}(\overline{c}_1^i, b_1^s, \overline{d}_1^j),$$

where $\overline{c}_1^i$ and $\overline{d}_1^j$, are arbitrary sequences over Q.

Sketch of the proof of $\overset{\circ}{1}$:

$$\overset{k}{A}(c_1^i, \boldsymbol{\alpha}, d_1^j) = \overset{l}{A}(c_1^i, \boldsymbol{\beta}, d_1^j) \Rightarrow$$

$$\overset{\bar{i}}{A}(a_1^{\bar{i}(n-1)}, \overset{k}{A}(c_1^i, \boldsymbol{\alpha}, d_1^j)) = \overset{\bar{i}}{A}(a_1^{\bar{i}(n-1)}, \overset{l}{A}(c_1^i, \boldsymbol{\beta}, d_1^j)) \overset{6.3VI}{\Rightarrow}$$

$$\overset{\bar{i}+k}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\alpha}, d_1^j) = \overset{\bar{i}+l}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\beta}, d_1^j) \Rightarrow$$

$$\overset{\bar{j}}{A}(\overset{\bar{i}+k}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\alpha}, d_1^j), b_1^{\bar{j}(n-1)}) =$$

$$\overset{\bar{j}}{A}(\overset{\bar{i}+l}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\beta}, d_1^j), b_1^{\bar{j}(n-1)}) \overset{6.3VI}{\Rightarrow}$$

$$\overset{\bar{i}+k+\bar{j}}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\alpha}, d_1^j), b_1^{\bar{j}(n-1)}) =$$

$$\overset{\bar{i}+l+\bar{j}}{A}(a_1^{\bar{i}(n-1)}, c_1^i, \boldsymbol{\beta}, d_1^j, b_1^{\bar{j}(n-1)}).$$

Sketch of the proof of $\overset{\circ}{2}$:

$\overset{k}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\alpha}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}) = \overset{l}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\beta}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}) \overset{6.3VI}{\Longrightarrow}$

$\overset{k-\bar{i}}{A}(\overset{\bar{i}}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\alpha}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}) =$

$\overset{l-\bar{i}}{A}(\overset{\bar{i}}{A}(c_1^{\bar{i}(n-1)+1}, \widehat{c}_1^{i-1}, \boldsymbol{\beta}, d_1^{\bar{j}(n-1)+1}, \widehat{d}_1^{j-1}) \overset{6.3VI}{\Longrightarrow}$

$\overset{k-\bar{i}-\bar{j}}{A}(\overset{\bar{i}}{A}(c_1^{\bar{i}(n-1)+1}), \widehat{c}_1^{i-1}, \boldsymbol{\alpha}, \overset{\bar{j}}{A}(d_1^{\bar{j}(n-1)+1}), \widehat{d}_1^{j-1}) =$

$\overset{l-\bar{i}-\bar{j}}{A}(\overset{\bar{i}}{A}(c_1^{\bar{i}(n-1)+1}), \widehat{c}_1^{i-1}, \boldsymbol{\beta}, \overset{\bar{j}}{A}(d_1^{\bar{j}(n-1)+1}), \widehat{d}_1^{j-1}) \Rightarrow$

$\overset{k-\bar{i}-\bar{j}}{A}(c, \widehat{c}_1^{i-1}, \boldsymbol{\alpha}, d, \widehat{d}_1^{j-1}) = \overset{l-\bar{i}-\bar{j}}{A}(c, \widehat{c}_1^{i-1}, \boldsymbol{\beta}, d, \widehat{d}_1^{j-1}),$

where $c = \overset{\bar{i}}{A}(c_1^{\bar{i}(n-1)+1})$ and $d = \overset{\bar{j}}{A}(d_1^{\bar{j}(n-1)+1})$.

Sketch of the proof of $\overset{\circ}{3}$:

a) $j \geq 2: \quad \overset{k}{A}(c_1^i, \boldsymbol{\alpha}, d_1^j) = \overset{l}{A}(c_1^i, \boldsymbol{\beta}, d_1^j) \Rightarrow$

$A(e_1, \overset{k}{A}(c_1^i, \boldsymbol{\alpha}, d_1^j), e_2^{n-1}) = A(e_1, \overset{l}{A}(c_1^i, \boldsymbol{\beta}, d_1^j), e_2^{n-1}) \overset{6.3VI}{\Longrightarrow}$

$\overset{k+1}{A}(e_1, c_1^i, \boldsymbol{\alpha}, d_1^{j-2}, d_{j-1}^j, e_2^{n-1}) = \overset{l+1}{A}(e_1, c_1^i, \boldsymbol{\beta}, d_1^{j-2}, d_{j-1}^j, e_2^{n-1}) \overset{6.3VI}{\Longrightarrow}$

$\overset{k}{A}(e_1, c_1^i, \boldsymbol{\alpha}, d_1^{j-2}, A(d_{j-1}^j, e_2^{n-1})) = \overset{l}{A}(e_1, c_1^i, \boldsymbol{\beta}, d_1^{j-2}, A(d_{j-1}^j, e_2^{n-1})) \Rightarrow$

$\overset{k}{A}(\widehat{c}_1^{i+1}, \boldsymbol{\alpha}, \widehat{d}_1^{j-1}) = \overset{l}{A}(\widehat{c}_1^{i+1}, \boldsymbol{\beta}, \widehat{d}_1^{j-1}),$

where $\widehat{c}_1^{i+1} = e_1, c_1^i$ and $\widehat{d}_1^{j-1} = d_1^{j-2}, A(d_{j-1}^j, e_2^{n-1})$.

b) $j = 1: \quad \overset{k}{A}(c_1^i, a_1^t, d) = \overset{l}{A}(c_1^i, b_1^s, d) \Rightarrow$

$A(c, \overset{k}{A}(c_1^i, a_1^t, d), e_1^{n-3}, \mathbf{e}(d, e_1^{n-3})) =$

$A(c, \overset{l}{A}(c_1^i, b_1^s, d), e_1^{n-3}, \mathbf{e}(d, e_1^{n-3})) \overset{6.3VI}{\Longrightarrow}$

$\overset{k+1}{A}(c, c_1^i, a_1^t, d, e_1^{n-3}, \mathbf{e}(d, e_1^{n-3})) =$

$\overset{l+1}{A}(c, c_1^i, b_1^s, d, e_1^{n-3}, \mathbf{e}(d, e_1^{n-3})) \overset{6.3VI}{\Longrightarrow}$

$\overset{k}{A}(c, c_1^i, a_1^{t-1}, A(a_t, d, e_1^{n-3}, \mathbf{e}(d, e_1^{n-3}))) =$

$\overset{l}{A}(c, c_1^i, b_1^{s-1}, A(b_s, d, e_1^{n-3}, \mathbf{e}(d, e_1^{n-3}))) \overset{2.6II}{\Longrightarrow}$

$\overset{k}{A}(c, c_1^i, a_1^{t-1}, a_t) = \overset{l}{A}(c, c_1^i, b_1^{s-1}, b_s) \Rightarrow$

$\overset{k}{A}(\widehat{c}_1^{i+1}, a_1^t, \emptyset) = \overset{l}{A}(\widehat{c}_1^{i+1}, b_1^s, \emptyset).$

where $\widehat{c}_{i+1} = c, c_1^i$.

The proof of the second part of the statement is similar.

Sketch of the proof of $\overset{\circ}{4}$:

$$\overset{k}{A}(c_1^i, a_1^t, d_1^j) = \overset{l}{A}(c_1^i, b_1^s, d_1^j) \Rightarrow$$
$$\overset{t}{A}(\overline{c}_1^i, \mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, \overset{k}{A}(c_1^i, a_1^t, d_1^j)) =$$
$$\overset{t}{A}(\overline{c}_1^i, \mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, \overset{l}{A}(c_1^i, b_1^s, d_1^j)) \overset{6.3VI}{\Longrightarrow}$$
$$\overset{t+k}{A}(\overline{c}_1^i, \mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, c_1^i, a_1^t, d_1^j) =$$
$$\overset{t+l}{A}(\overline{c}_1^i, \mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, c_1^i, b_1^s, d_1^j) \overset{6.3IV}{\Longrightarrow}$$
$$\overset{k}{A}(\overline{c}_1^i, \overset{t}{A}(\mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, c_1^i, a_1), a_2^t, d_1^j) =$$
$$\overset{l}{A}(\overline{c}_1^i, \overset{t}{A}(\mathsf{E}(e_{i+1}^{t(n-1)-1}, c_1^i), e_{i+1}^{t(n-1)-1}, c_1^i, b_1), b_2^s, d_1^j) \overset{2.1II,6.4VI}{\Longrightarrow}$$
$$\overset{k}{A}(\overline{c}_1^i, a_1, a_2^t, d_1^j) = \overset{l}{A}(\overline{c}_1^i, b_1, b_2^s, d_1^j) \Leftrightarrow$$
$$\overset{k}{A}(\overline{c}_1^i, a_1^t, d_1^j) = \overset{l}{A}(\overline{c}_1^i, b_1^s, d_1^j),$$

where E is an $\{1, t(n-1)+1\}-$neutral operation of $(t(n-1)+1)-$ group; $1 \leq i \leq t(n-1)-1$.

The proof of the second part of the statement is similar.

Sketch of the proof of (ii) :

$$\emptyset\ \theta\ \emptyset \overset{(2)}{\Leftrightarrow} A(a_1^i, \emptyset, a_{i+1}^n) = A(a_1^i, \emptyset, a_{i+1}^n)$$
$$\Leftrightarrow A(a_1^i) = A(a_1^i),$$

$i \in \{0, \ldots, n\}$; for $i = 0$, $a_1^i = \emptyset$, and for $i = n$, $a_{i+1}^n = \emptyset$.

Sketch of the proof of (iii) :

$$\overset{t}{A}(\boldsymbol{\alpha}, y) = y \overset{6.4VI}{\Longleftrightarrow} \overset{t}{A}(\boldsymbol{\alpha}, \overset{k}{A}(x_1^{k(n-1)+1})) = \overset{k}{A}(x_1^{k(n-1)+1})$$
$$\overset{6.3VI}{\Longleftrightarrow} \overset{t+k}{A}(\boldsymbol{\alpha}, x_1^{k(n-1)+1}) = \overset{k}{A}(x_1^{k(n-1)+1})$$
$$\overset{(i)}{\Longleftrightarrow} \overset{t+k}{A}(\emptyset, \boldsymbol{\alpha}, x_1^{k(n-1)+1}) = \overset{k}{A}(\emptyset, \emptyset, x_1^{k(n-1)+1})$$
$$\overset{(2)}{\Longleftrightarrow} \boldsymbol{\alpha}\ \theta\ \emptyset.$$

An example of a nonempty sequence which is equivalent to the empty sequence is:

$$\overset{(1)}{a}{}_1^{n-2}, \overset{(2)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(2)}{a}{}_1^{n-2}), \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}).$$

Indeed:
$\overset{2}{A}(y, \overset{(1)}{a}{}_1^{n-2}, \overset{(2)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(2)}{a}{}_1^{n-2}), \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}))\overset{6.3VI}{=}$
$A(y, \overset{(1)}{a}{}_1^{n-3}, A(\overset{(1)}{a}{}_{n-2}, \overset{(2)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(2)}{a}{}_1^{n-2})), \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}))\overset{3.4III}{=}$
$A(y, \overset{(1)}{a}{}_1^{n-3}, \overset{(1)}{a}{}_{n-2}, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2})) =$
$A(y, \overset{(1)}{a}{}_1^{n-2}, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}))\overset{2.1II}{=} y.$

Sketch of the proof of (iv) :

$a)$ $\boldsymbol{\alpha}\ \theta\ \boldsymbol{\alpha} \overset{(2)}{\Longleftrightarrow} \overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}).$

$b)$ $\boldsymbol{\alpha}\ \theta\ \boldsymbol{\beta} \overset{(2)}{\Longleftrightarrow} (\exists k \in N)(\exists l \in N)(\exists \boldsymbol{\gamma} \in \Gamma)(\exists \boldsymbol{\delta} \in \Gamma)\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta})$
$\Longleftrightarrow (\exists k \in N)(\exists l \in N)(\exists \boldsymbol{\gamma} \in \Gamma)(\exists \boldsymbol{\delta} \in \Gamma)\overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}) = \overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta})$
$\overset{(2)}{\Longleftrightarrow} \boldsymbol{\beta}\ \theta\ \boldsymbol{\alpha}.$

$c)$ There exsist $k, l, \bar{l}, t, \bar{t} \in N$ and $\boldsymbol{\delta}, \overline{\boldsymbol{\delta}}, \boldsymbol{\varphi}, \overline{\boldsymbol{\varphi}} \in \Gamma$ such that the following sequence of implications equivalences holds:

$\boldsymbol{\alpha}\ \theta\ \boldsymbol{\beta} \wedge \boldsymbol{\beta}\ \theta\ \boldsymbol{\gamma} \overset{(2)}{\Longleftrightarrow}$
$\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) \wedge \overset{\bar{l}}{A}(\overline{\boldsymbol{\delta}}, \boldsymbol{\beta}, \overline{\boldsymbol{\varphi}}) = \overset{\bar{t}}{A}(\overline{\boldsymbol{\delta}}, \boldsymbol{\gamma}, \overline{\boldsymbol{\varphi}}) \overset{\mathring{1}-\mathring{4}}{\Longrightarrow}$
$\overset{\widehat{k}}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\alpha}, \widehat{\boldsymbol{\varphi}}) = \overset{\widehat{l}}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\beta}, \widehat{\boldsymbol{\varphi}}) \wedge \overset{\widehat{l}}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\beta}, \widehat{\boldsymbol{\varphi}}) = \overset{t}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\gamma}, \widehat{\boldsymbol{\varphi}}) \Rightarrow$
$\overset{\widehat{k}}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\alpha}, \widehat{\boldsymbol{\varphi}}) = \overset{t}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\gamma}, \widehat{\boldsymbol{\varphi}}) \overset{(2)}{\Longleftrightarrow}, \boldsymbol{\alpha}\ \theta\ \boldsymbol{\gamma}.$

$d)$ There exist $k, \overline{k}, \widehat{k}, l, \bar{l}, \widehat{l}, t, \bar{t}, \widehat{t} \in N$ and $\boldsymbol{\gamma}, \overline{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\gamma}}, \boldsymbol{\delta}, \overline{\boldsymbol{\delta}}, \widehat{\boldsymbol{\delta}} \in \Gamma$ such that the following sequence of implications (equivalences) holds:
$\boldsymbol{\alpha}\ \theta\ \overline{\boldsymbol{\alpha}} \wedge \boldsymbol{\beta}\ \theta\ \overline{\boldsymbol{\beta}} \overset{(2)}{\Longleftrightarrow}$
$\overset{\bar{k}}{A}(\overline{\boldsymbol{\gamma}}, \boldsymbol{\alpha}, \overline{\boldsymbol{\delta}}) = \overset{\bar{l}}{A}(\overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\delta}}) \wedge \overset{\widehat{k}}{A}(\widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \widehat{\boldsymbol{\delta}}) = \overset{\widehat{l}}{A}(\widehat{\boldsymbol{\gamma}}, \overline{\boldsymbol{\beta}}, \widehat{\boldsymbol{\delta}}) \overset{\mathring{1}-\mathring{4}}{\Longrightarrow}$
$\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{\delta}) \wedge \overset{l}{A}(\boldsymbol{\gamma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{\delta}) = \overset{t}{A}(\boldsymbol{\gamma}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \boldsymbol{\delta}) \Rightarrow$
$\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}) = \overset{t}{A}(\boldsymbol{\gamma}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \boldsymbol{\delta}) \Rightarrow$
$\overset{(2)}{\Longleftrightarrow}, \boldsymbol{\alpha} * \boldsymbol{\beta}\ \theta\ \overline{\boldsymbol{\alpha}} * \overline{\boldsymbol{\beta}}.$

Sketch of the proof of (v) :

Let

$$\boldsymbol{\alpha} = \overline{\overset{(j)}{a}{}_1^{n-2}}\Big|_{j=1}^{t}, b_1^i \ (t \in N \cup \{0\},\ 0 \le i < n-2)$$

be a sequence over Q. Also, let $\mathbf{e}$ be $\{1,n\}-$neutral operation of the $n-$grup $(Q;A)$. Further on, let

$$\boldsymbol{\varepsilon} = \overline{\overset{(j)}{a}{}_1^{n-2}}\Big|_{j=1}^{t}, b_1^i, c_{i+1}^{n-2}, \mathbf{e}(b_1^i, c_{i+1}^{n-2}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}).$$

Then $\boldsymbol{\varepsilon}\ \theta\ \emptyset$.

2. Proposition *[Post 1940]: Let $n \ge 3$, let $(Q;A)$ be an $n-$grup and let $(\Gamma;*)$ be the semigroup from Prop. 1. Also, let θ be the congruence relation in $(\Gamma;*)$ [from Prop. 1]. Finally, let for all $C(\boldsymbol{\alpha}), C(\boldsymbol{\beta}) \in \Gamma/\theta$*

(3) $\quad C(\boldsymbol{\alpha}) \cdot C(\boldsymbol{\beta}) \overset{def}{=} C(\boldsymbol{\alpha} * \boldsymbol{\beta}).$

Then $(\Gamma/\theta;\cdot)$ grupa.

Proof. 1) $(\Gamma/\theta;\cdot)$ is a semigroup /since $(\Gamma;*)$ is a semigroup/.

2) $C(\boldsymbol{\alpha}) \cdot C(\emptyset) \overset{(3)}{=} C(\boldsymbol{\alpha} * \emptyset) \overset{1(i)}{=} \boldsymbol{\alpha}$.

3) By the proof of 1-(v) we have

$$\begin{aligned} C(\emptyset) &\overset{1(v)}{=} C(\boldsymbol{\varepsilon}) \\ &\overset{(1)}{=} C(\boldsymbol{\alpha} * c_{i+1}^{n-2}, \mathbf{e}(b_1^i, c_{i+1}^{n-2}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, (\overset{(1)}{a}{}_1^{n-2})) \\ &\overset{(3)}{=} C(\boldsymbol{\alpha}) \cdot C(c_{i+1}^{n-2}, \mathbf{e}(b_1^i, c_{i+1}^{n-2}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, (\overset{(1)}{a}{}_1^{n-2})). \end{aligned}$$

Whence, $C(c_{i+1}^{n-2}, \mathbf{e}(b_1^i, c_{i+1}^{n-2}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, (\overset{(1)}{a}{}_1^{n-2}))$ is the right inverse element of the element $\boldsymbol{\alpha}$ with respect to the right neutral element $C(\emptyset)$ of $(\Gamma/\theta;\cdot)$.

4) Finally, by 1)-3), we conclude that Prop.2 holds. □

3. Theorem *[Post 1940]: Let $n \ge 3$, let $(Q;A)$ be an $n-$grup and $(\mathbf{\Gamma};\cdot)$ /$\overset{def}{=}(\Gamma/\theta;\cdot)$ / the group from 2. Also, let $(\mathbf{\Gamma};\mathbf{A})$ be the $n-$grup defined by*

(4) $\mathbf{A}(\alpha_1^n) \overset{def}{=} \alpha_1 \cdot \ldots \cdot \alpha_n$

for every $\alpha_1^n \in \mathbf{\Gamma}$. Then there is an $n-$subgroup $(\mathbf{Q};\mathbf{A})$ of the $n-$grup $(\mathbf{\Gamma};\mathbf{A})$ such that $(\mathbf{Q};\mathbf{A})$ and $(Q;A)$ are isomorphic.

Remark: We say that the group $(\mathbf{\Gamma};\cdot)$ is a **covering grup** *of an $n-$grup $(Q;A)$. Furthermore, Th. 3 has also the following formulation: Every $n-$grup*

$(n \geq 3)$ *has a covering group.*

Proof. Firstly we prove the following statements:

$^\circ 1$ $(\mathbf{\Gamma}; \mathbf{A})$ is an $n-$semigroup;

$^\circ 2$ There exist bijective mapping F of the set Q to the set $\mathbf{Q} \overset{def}{=} \{C(x) | x \in Q\}$ *[*$C(x) \in \Gamma/\theta$ *]* such that for all $x_1^n \in Q$ the following equality holds

$FA(x_1^n) = \mathbf{A}(F(x_1), \ldots, F(x_n))$; and

$^\circ 3$ The $n-$grupoid $(\mathbf{Q}; \mathbf{A})$ *[*$^\circ 2$*]* has an $\{1, n\}-$neutral operation E such that for all $a_1^{n-2} \in Q$ the following equality holds

$F\mathbf{e}(a_1^{n-2}) = \mathsf{E}(F(a_1), \ldots, F(a_{n-2}))$,
where $\mathbf{e}$ is an $\{1, n\}-$neutral operation of the $n-$group $(Q; A)$ and F is from $^\circ 2$.

Proof of $^\circ 1$: By (4) and by Prop.2.

Proof of $^\circ 2$

1) Every $\mathbf{a} \in \Gamma/\theta$ has at least one sequence of length one.

Indeed:

$a)$ $y \in C(x) \Leftrightarrow y \; \theta \; x$
$\overset{(2)}{\Leftrightarrow} (\exists k \in N)(\exists \boldsymbol{\alpha} \in \Gamma)(\exists \boldsymbol{\beta} \in \Gamma) \overset{k}{A}(\boldsymbol{\alpha}, y, \boldsymbol{\beta}) = \overset{k}{A}(\boldsymbol{\alpha}, x, \boldsymbol{\beta})$; and

$b)$ $\overset{k}{A}(\boldsymbol{\alpha}, y, \boldsymbol{\beta}) = \overset{k}{A}(\boldsymbol{\alpha}, x, \boldsymbol{\beta}) \overset{6.4VI}{\Rightarrow} y = x$.

2) Let
$$F(x) \overset{def}{=} C(x)$$
for all $x \in Q$. Then, by 1), F is a bijection from the set Q to the set $\mathbf{Q}$.

3) Let
$$\mathbf{A}(C(a_1), \ldots, C(a_n)) \overset{(4)}{=} C(a_1) \cdot \ldots \cdot C(a_n)$$
for all $a_1^n \in Q$. Then
$$FA(a_1^n) = \mathbf{A}(F(a_1), \ldots, F(a_n))$$
for all $a_1^n \in Q$.

Indeed:

$\overline{a})$ $b = A(a_1^n) \overset{1.1I}{\Longleftrightarrow} A(b, x_1^{n-1}) = A(A(a_1^n), x_1^{n-1})$

$\overset{6.3VI}{\Longleftrightarrow} A(b, x_1^{n-1}) = \overset{2}{A}(a_1^n, x_1^{n-1})$

$\Longleftrightarrow A(\emptyset, b, x_1^{n-1}) = \overset{2}{A}(\emptyset, a_1^n, x_1^{n-1})$

$\overset{(2)}{\Longleftrightarrow} b\ \theta\ a_1^n.$

$\overline{b})$ $C(b) \overset{\overline{a})}{=} C(a_1^n)$

$\overset{(3)}{=} C(a_1^n) \cdot \ldots \cdot C(a_n).$

$\overline{c})$ $C(A(a_1^n)) \overset{\overline{a}),\overline{b})}{=} C(a_1^n) \cdot \ldots \cdot C(a_n).$

By $\overline{c}), 2)$ and (4), we have

$$FA(a_1^n) = \mathbf{A}(F(a_1), \ldots, F(a_n))$$

for all $a_1^n \in Q$.

Proof of $^\circ 3$:

Let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of the $n-$grup $(Q; A)$. By 3.4-III, we have

(a) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ and

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$

for all $x, a_1^{n-2} \in Q$. Whence, by $^\circ 2$, we conclude that the following equalities hold

$(\overline{a})$ $= \mathbf{A}(F(x), F(a_1), \ldots, F(a_{n-2}), F\mathbf{e}(a_1^{n-2})) = F(x)$

$(\overline{b})$ $\mathbf{A}(F\mathbf{e}(a_1^{n-2}), F(a_1), \ldots, F(a_{n-2}), F(x)) = F(x)$

for all $x, a_1^{n-2} \in Q$.

In addition (by F bijection) let

$$\mathsf{E}(b_1^{n-2}) \overset{def}{=} F\mathbf{e}(F^{-1}(b_1), \ldots, F^{-1}(b_{n-2}))$$

for all $b_1^{n-2} \in \mathbf{Q}$, i.e.

(5) $F\mathbf{e}(a_1^{n-2}) = \mathsf{E}(F(a_1), \ldots, F(a_{n-2}))$

for all $a_1^{n-2} \in Q$. Furthermore, by the substitution of (5) in $(\overline{a})$ and $(\overline{b})$, we obtain

$\mathbf{A}(y, b_1^{n-2}, \mathsf{E}(b_1^{n-2}) = y$ and

$\mathbf{A}(\mathsf{E}(b_1^{n-2}), b_1^{n-2}, y) = y;$

$y = F(x),\ b_1 = F(a_1), \ldots, b_{n-2} = F(a_{n-2}).$

Finally, by Th. 2.2-IX and by $°1 -° 3$, we conclude that Th.3 holds. □

4. Remark: *Let* $\boldsymbol{\alpha}\ \theta\ \boldsymbol{\beta}$. *Then* $||\boldsymbol{\alpha}| - |\boldsymbol{\beta}|| = t(n-1)$, *where* $t \in N \cup \{0\}$.

Indeed:

By (2) *from Prop.1, we have*

$$\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}),$$
$$|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| = k(n-1) + 1 \ \textit{and}$$
$$|\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = l(n-1) + 1.$$

Also let $k \geq l$. *Then we obtain:*

$$|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| - |\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = (k-l)(n-l) \ \textit{and}$$
$$|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| - |\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = |\boldsymbol{\gamma}| + |\boldsymbol{\alpha}| + |\boldsymbol{\delta}| - (|\boldsymbol{\gamma}| + |\boldsymbol{\beta}| + |\boldsymbol{\delta}|)$$
$$= |\boldsymbol{\alpha}| - |\boldsymbol{\beta}|.$$

Finally, whence we conclude that the following equality holds

$$= ||\boldsymbol{\alpha}| - |\boldsymbol{\beta}|| = t(n-1),$$

where $t \in N \cup \{0\}$.

5. Proposition *[Post 1940]: Let* $n \geq 3$, $(Q; A)$ *be an* $n-$*grup and* $(\mathbf{\Gamma}; \cdot)$ *[*$\overset{def}{=} (\Gamma/\theta; \cdot)$ *] be a group from Prop.2 [coverning group of an* $n-$*group* $(Q; A)$*]. Also, let*

$$\mathbf{H} \overset{def}{=} \{\mathbf{a} | \mathbf{a} \subseteq \Gamma\ \mathbf{a} \ni a \wedge\ |a| = t(n-1)\ \wedge\ t \in N \cup \{0\}\}.$$

Then $(\mathbf{H}; \cdot)$ *is a normal subgroup of the group* $(\mathbf{\Gamma}; \cdot)$. *Moreover, for all* $\mathbf{a} \in \mathbf{Q}$ *the following equality holds*

$$\mathbf{aH} = \mathbf{Q}.$$

Sketch of the proof.

a) $\overline{\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{k}} * \overline{\overset{(j)}{b}{}_1^{n-2}\Big|_{j=1}^{l}} = \overline{\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{k}, \ \overset{(j)}{b}{}_1^{n-2}\Big|_{j=1}^{l}}$ and
$|\overline{\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{k}}, \ \overline{\overset{(j)}{b}{}_1^{n-2}\Big|_{j=1}^{l}}| = (k+l)(n-1).$

b) Let $\boldsymbol{\varepsilon}$ be the neutral element of the group $(\boldsymbol{\Gamma};\cdot)$. Then, for example

$$\overline{\overset{(i)}{a}{}_1^{n-2}}\Big|_{i=1}^{t}, \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2});$$

see (*iii*). Whence, by

$$|\overline{\overset{(i)}{a}{}_1^{n-2}}\Big|_{i=1}^{t}, \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2})| = t\cdot(n-1),$$

we conclude that $\boldsymbol{\varepsilon}$ v $\mathbf{H}$.

c) Let

$$\overline{\overset{(i)}{a}{}_1^{n-2}}\Big|_{i=1}^{t}, b_1^t \quad (t \in N \cup \{0\})$$

be an arbitrary sequence from arbitrary $\mathbf{a} \in \mathbf{H}$.
Then,

- for $t < n-2$
$$\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{t}, b_1^t, c_{t+1}^{n-2}, \mathbf{e}(b_1^t, c_{t+1}^{n-2}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}) \in \boldsymbol{\varepsilon},$$

- for $t = n-2$
$$\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{t}, b_1^t, \mathbf{e}(b_1^t,)\mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}) \in \boldsymbol{\varepsilon} \text{ and}$$

- for $t > n-2$
$$\overset{(i)}{a}{}_1^{n-2}\Big|_{i=1}^{t}, b_1^t, b_{t+1}^{k(n-2)}, \mathbf{e}(b_1^{n-2}), \ldots, \mathbf{e}(b_{(k-1)(n-2)}^{k(n-2)}), \mathbf{e}(\overset{(t)}{a}{}_1^{n-2}), \ldots, \mathbf{e}(\overset{(1)}{a}{}_1^{n-2}) \in \boldsymbol{\varepsilon}.$$

d) By $a) - c)$, we conclude that $(\mathbf{H};\cdot)$ is a grup.

e) Let h be an arbitrary sequence from an arbitrary $\mathbf{h} \in \mathbf{H}$. Also, let a be an arbitrary sequence from an arbitrary $\mathbf{a} \in \boldsymbol{\Gamma}$, and $a^{-1} \in \mathbf{a}^{-1} \in \boldsymbol{\Gamma}$. Then:

$$|a\ h\ a^{-1}| \in \{t(n-1)|t \in N \cup \{0\}.$$

Finally, by propositions 1-5 and the proof of 1-5, we obtain:

6. Theorem *[Post 1940]: Let* $n \geq 3$, *let* $(Q;A)$*be an* $n-$*group,* $(\boldsymbol{\Gamma};\cdot)$ *its covering group and* $(\mathbf{H};\cdot)$ *a normal subgroup of the group* $(\mathbf{H};\cdot)$ *Then:*$(\boldsymbol{\Gamma}/\mathbf{H};)$

is a finite cyclic group,$\mathbf{Q}$ *its generator,* $|\mathbf{\Gamma}/\mathbf{H}|\ |(n-1)$ *and for every* $x_1^n \in Q, A(x_1^n) = x_1 \cdot \ldots \cdot x_n$.

Remark: For the proofs of Th.6 see also in books: [Bruck 1958], [Belousov 1972] in [Gal'mak 2003]. Cf. also the book [Kurosh 1974].

REFERENCES

[Belousov 1965] V.D.Belousov, Systems of quasigroups with generalized identities, (Russian), Usp.mat.nauk, 20(1965) No.1,75-146.

[Belousov 1967] V.D.Belousov, Foundation of the theory of quasigoups and loops,(Russian), "Nauka", Moscow 1967.

[Belousov 1972] V.D.Belousov, $n-$ary quasigroups, (Russian), Stiinta, Kishinev 1972.

[Belousov, Sandik 1966] V.D.Belousov, M. D. Sandik, $n-$Ary quasigroups and Loops, (Russian), Sib. math. zh. Tom VII, No. 1 (1966), 31-54.

[Boccioni 1965] D. Boccioni, Caratterizzazione di una classe di anelli generalizzati, Rend. Sem. Mat. Univ. Padova 35 (1965), 116-127.

[Bruck 1946] R.H. Bruck, Contributions to the theory of loops, Trans. Amer. Math. Soc. 60 (1946), 245-354.

[Bruck 1971] R.H. Bruck, A survey of binary systems, Springer-Verlag, Berlin-Heidelberg-Götingen 1958.

[Celakoski 1977] N. Celakoski, On some axsiom system for $n-$groups, Mat.Bilten (Skopje) 27(1977), 5-14.

[Clifford, Preston 1964] A.H. Clifford and G.B. Preston, The algebraic theory of semigroup, Vol. I, Amer. Math. Society 1964.

[Cohn 1968] P.M. Cohn, Universal algebra,(Russian),"Mir", Moscow 1968.

[Crombez 1972/1] G. Crombez, On (n,m)-rings, Abh. Math. Sem. Univ. Hamburg 37(1972),180-199.

[Crombez 1972/2] G. Crombez, On partially ordered $n-$groups, Abh. Math. Sem. Univ. Hamburg 38(1972),141-146.

[Crombez, Six 1974] G. Crombez, G. Six, On topological n-groups, Abh. Math. Sem. Univ. Hamburg 41(1974),115-124.

[Čupona 1965] Ǵ. Čupona, On $[m, n]-$rings,(Macedonian), Bull. Soc. Math. Phys. R.S. Macedonie 16 (1965),5-10.

[Čupona 1969] Ǵ. Čupona, Finitary associative operations, (Serbo-Croatian), Mat. bibl. 39(1969),135-149.

[Čupona 1971] Ǵ. Čupona, On topological $n-$groups, Bull. Soc. Math. Phys. RSM 22(1971),5-10.

[Čupona 1983] Ǵ. Čupona, Vector valued semigroups, Semigroup Forum 26(1983), 65-74.

[Čupona, Trpenovski 1961] Ǵ. Čupona, B. Trpenovski, Finitary associative oprations with neutral element, Mat. Bilten (Skopje) 12(1961), 15-24.

[Čupona, Celakoski 1980] Ǵ. Čupona, N. Celakoski, Polyadic subsemigroups of semigroups, Algebraic Conference I, Collection of papers, Skopje 1980, 131-152.

[Čupona, Celakoski, Markovski, Dimovski 1988] Ǵ. Čupona, N. Celakoski, S. Markovski and D. Dimovski, Vector valued groupoids, semigroups and groups, in: Vector vauled semigroups and groups, (B. Popov, Ǵ. Čupona and N. Celakoski, eds.), Skopje 1988, 1-78.

[Dickson 1905] L.E. Dickson, Definitions of a group and a field by independent postulates, Trans. Amer. Math. Soc. 6(1905),198-204.

[Dudek 1990] W.A. Dudek, On some classes of $n-$groups, (Russian), Stiinta, Kishinev 1990.

[Dudek 1995] W.A. Dudek, Varieties of polyadic groups, Filomat (Niš) 9(1995) No 3,657-674.

[Dudek 2001] W.A. Dudek, On some old and new problems in $n-$ary groups, Quasigroups and Related Systems 8 (2001), 15-36.

[Dudek, Glazek, Gleichgewicht 1977] W.A. Dudek, K. Glazek, B. Gleichgewicht, A note on the axiom of $n-$groups, Coll. Math. Soc. J. Bolyai, 29. Universal Algebra, Esztergom (Hungary), 1977, 195-202.

[Dudek, Michalski 1982] W.A. Dudek and J. Michalski, On a generalization of Hosszú Theorem, Demonstratio Math. 15(1982), 783-805.

[Dörnte 1928] W. Dörnte, Untersuchengen über einen verallgemeinerten Gruppenbegriff, Math. Z. 29(1928), 1-19.

[Enders 1995] N. Enders, On topological $n-$groups and their correspondig groups, Discussiones Math.-Algebra and Stohastic Methods 15(1995),163-169.

[Fuchs 1963] L. Fuchs, Partially ordered algebraic systems, Pergamon Press, Oxford 1963.

[Furstenberg 1955] H. Furstenberg, The inverse operation in groups, Proc. Amer. Math. Soc. 6(1955), 991-997.

[Gal'mak 2000] A.M. Gal'mak, Remarks on polyadic groups, Quasigroups and Related Systems 7(2000), 67-71.

[Gal'mak 2003] A.M. Gal'mak, $n-$ary groups. Part 1, (Russian), Gomel' 2003.

[Gal'mak, Vorob'ev 1998] A.M. Gal'mak and G.N. Vorob'ev, Ternary groups of transformations, (Russian), Minsk, "Belaruskaya Navuka" 1998.

[Gleichgewicht, Glazek 1967] B. Gleichgewicht, K. Glazek, Remarks on $n-$groups as abstract algebras, Colloq. Math. 17(1967), 209-219.

[Glazek 1982] K. Glazek, Bibliography of $n-$groups (polyadic groups) and some group-like $n-$ary systems, Proc of the Symposium "$n-$ary structures" (Skopje 1982), Macedonian Academy of Sciences and Arts, Skopje 1982, 253-289.

[Gluskin 1965] L.M. Gluskin, Position operatives, (Russian), Mat. Sb., t. 68(110) (1965) No. 3, 444-472.

[Higman, Neuman 1952] G. Higman, B.H. Neuman, Groups as groupoids with one law, Publ. Math. Debrecen 2(1952), 215-221.

[Hosszú 1963] M. Hosszú, On the explicit form of $n-$group operatin, Publ. Math. Debrecen 10(1963), 88-92.

[Hughes 1957] N.J.S. Hughes, A theorem on isoptic groupoids, J. London Math. Soc. 32(1957), 510-511.

[Iancu 1999] L. Iancu, Contributions to the theory of $(m, n)-$rings and $n-$modules, PhD Thesis, University "Babes-Bolyai" of Cluj-Napoca, 1999.

[Janeva 1995] B. Janeva, Congruences on $m-$groups, Mat. Blten (Skopje) 45(1995), 85-90.

[Kurosh 1962] A.G. Kurosh, Lectures of general algebra, (Russian), Moskow 1962.

[Kurosh 1967] A.G. Kurosh, The theory of groups, (Russian), "Nauka", Moscow 1967.

[Kurosh 1974] A.G. Kurosh, General algebra (Lectures 1969-1970), (Russian), "Nauka", Moscow 1974.

[Michalski 1979] J. Michalski, On some functors from the category of $n-$groups, Bull. Pol. Acad. Sci., Ser. Sci. Mat. Astr. et Phys. 27(1979), 437-441.

[Monk, Sioson 1971] J.D. Monk, F.M. Sioson, On the general theory of

$m-$groups, Fund. Math. 72(1971), 233-244.

[Movsisyan 1986] Yu. M. Movsisyan, Introduction to the theory of algebras with hyperidentities, (Russian), Izdat. Erevan Univ., Erevan 1986.

[Paunić 1984] Dj. Paunić, Localization in $(m, n)-$rings, Proceedings of the conference "Algebra and logic - Zagreb 1984", Univ. of Novi Sad, Novi Sad 1985, 123-133.

[Pontryagin 1973] L.S. Pontryagin, Topological groups, (Russian), "Nauka", Moscow 1973.

[Post 1940] E.L. Post, Polyadic groups, Trans. Amer. Math. Soc. 48(1940), 208-350.

[Rusakov 1992] S.A. Rusakov, Algebraic $n-$ary Systems, (Russian), "Navuka i tehnika", Minsk 1992.

[Sokhatsky 1998] F.M. Sokhatsky, Invertible elements in associates and semigroups 1, Quasigroups and Related Systems 5(1998), 53-68.

[Sokhatsky, Yurevych 1999] F.M. Sokhatsky and O. Yurevich, Invertible elements in associates and semigroups 2, Quasigroups and Related Systems 6(1999), 61-70.

[Sokolov 1976] E.I. Sokolov, On the Gluskin-Hosszú theorem for Dörnte $n-$groups, (Russian), Mat. Issled. 39(1976), 187-189.

[Tasić 1988] V. Tasić, On single-law definitions of groups, Bull. Austral. Math. Soc. 37(1988), 101-106.

[Tvermoes 1953] H. Tvermoes, Über eine Verallgemeinrung des Gruppenbegriffs, Math. Scand. 1(1953), 18–30.

[Tyutin 1985] V.I. Tyutin, On the axiomatics of $n-$ary groups, Doklady Akademii Nauk BSSR 29(1985) No.8, 691-693.

[Ušan 1988] J. Ušan, Neutral operations of $n-$groupoids, (Russian), Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 18(1988) No.2, 117-126.

[Ušan 1989] J. Ušan, Neutral operations of $(n, m)-$groupoids, (Russian), Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 19(1989) No.2, 125-137.

[Ušan 1994] J. Ušan, A comment on $n-$groups, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 24(1994) No.1, 281-288.

[Ušan 1995/1] J. Ušan, On Hosszú-Gluskin algebras corresponding to the same $n-$group, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 25(1995) No.1, 101-119.

[Ušan 1995/2] J. Ušan, On $n-$groups with $\{i, j\}-$neutral operation for $\{i, j\} \neq \{1, n\}$, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 25(1995) No.2, 167-178.

[Ušan 1997/1] J. Ušan, On congruences on $n-$groups, Novi Sad, J. Math. 27(1997) No.2, 89-100.

[Ušan 1997/2] J. Ušan, $n-$groups as variety of type $< n, n-1, n-2 >$, in: Algebra and Model Theory, (A.G. Pinus and K.N. Ponomaryov, eds.), Novosibirsk, 1997, 182-208.

[Ušan 1997/3] J. Ušan, On $n-$groups, Maced. Acad. Sci. and Arts, Contributions, Sect. Math. Techn. Sci. XVIII1-2(1997), 17-20.

[Ušan 1997/4] J. Ušan, $n-$groups as $n-$groupoids with laws, Quasigroups and Related Systems, 4(1997), 67-76.

[Ušan 1998/1] J. Ušan, On congruence classes of $n-$groups, Novi Sad J. Math. 28(1998) No.1, 75-89.

[Ušan 1998/2] J. Ušan, Congruences of $n-$group and of associated Hosszú-Gluskin algebras, Novi Sad J. Math. 28(1998) No.2, 91-108.

[Ušan 1998/3] J. Ušan, On $(k(n-1)+1)-$semigroups with $(n-2)-$neutral operations, Mat. Bilten Skopje 22(1998), 5-14.

[Ušan 1998/4] J. Ušan, On topological $n-$groups, Math. Moravica 2(1998), 149-159.

[Ušan 1999/1] J. Ušan, A note on topological $n-$groups, Math. Moravica 3(1999), 111-115.

[Ušan 1999/2] J. Ušan, A comment of power $n-$group, Math. Moravica 3(1999), 117-126.

[Ušan 1999/3] J. Ušan, Note on $(n, m)-$groups, Math. Moravica 3(1999), 127-139.

[Ušan 1999/4] J. Ušan, A comment on a class of polyadic groups, Scientific Bulletin of the Polytechnic University Timişoara, Math-Phys. Ser. 44(58)-2(1999), 8-14.

[Ušan 1999/5] J. Ušan, The $n-$ary case of a Bruck's Theorem, Scientific Bulletin of the Polytechnic University Timişoara, Math-Phys. Ser. 44(58)-2(1999), 46-50.

[Ušan 1999/6] J. Ušan, On $TS - n-$groups, Quasigroups and Related Systems 6(1999), 81-86.

[Ušan 1999/7] J. Ušan, A note on $n-$groups for $n \geq 3$, Novi Sad J. Math. 29(1999) No.1, 55-59.

[Ušan 2000] J. Ušan, On $(n, m)-$groups, Math. Moravica 4(2000), 115-118.

[Ušan 2001/1] J. Ušan, Description of super-associative algebras with $n-$quasigroup operations, Math. Moravica 5(2001), 129-157.

[Ušan 2001/2] J. Ušan, A comment on $(n, m)-$groups for $n \geq 3m$, Math. Moravica 5(2001), 159-162.

[Ušan 2002/1] J. Ušan, One characterization of near-P-polyagroup, Math. Moravica Vol. 6(2002), 127-130.

[Ušan 2002/2] J. Ušan, Note on congruence classes of $n-$groups, Math. Moravica Vol. 6(2002), 131-135.

[Ušan 2005/1] J. Ušan, $(n, m)-$groups in the light of the neutral operations, preprint 2005.

[Ušan 2005/2] J. Ušan, A survey of $NP-$polyagroups, preprint 2005.

[Ušan, Žižović 1997] J. Ušan and M. Žižović, On ordered $n-$groups, Quasigroups and Related Systems 4(1997),77-87.

[Ušan, Žižović 1999/1] J. Ušan and M. Žižović, On invaiant $n-$subgroups, Proceedings 8^{th} Symposium of Mathematics and its Applications, Timişoara 1999, 259-270.

[Ušan, Žižović 1999/2] J. Ušan and M. Žižović, Some remarks on $(m, n)-$rings, Filomat (Niš), 13(1999), 53-57.

[Ušan, Žižović 2001] J. Ušan and M. Žižović, Note on near-P-polyagroups, Filomat (Niš) 15(2001), 85-90.

[Ušan, Žižović 2002/1] J. Ušan and M. Žižović, Note on n-groups, Math. Moravica Vol. 6(2002), 137-144.

[Ušan, Žižović 2002/2] J. Ušan and M. Žižović, Note on polyagroups, Math. Moravica Vol. 6(2002), 145-149.

[Ušan, Žižović 2004] J. Ušan and M. Žižović, On congruences of superassociative algebras with $n-$quasigroup operations, $n \geq 3$, Math. Moravica 8-2(2004), 95-99.

[Ušan, Galić 2000] J. Ušan and R. Galić, $(m, n)-$Rings as algebras with only one operations, Mathematical Communications 5(2000) No. 2, 133-141.

[Ušan, Galić 2001] J. Ušan and R. Galić, On $NP-$polyagroups, Math.

Comm. 6(2001) No.2, 153-159.

[Žižović 1976] M. Žižović, Topological analogy of Hosszú-Gluskin's theorem, (Serbo-Croatian), Mat. vesnik 13(28)(1976), 233-235.

[Žižović 1998] M. Žižović, On $\{1, n\}-$neutral, inversing and skew operations of $n-$groups, Mathematica Moravica 2(1998), 169-173.

[Žižović, Kočinac 1988] M. Žižović, Lj. Kočinac, Some remarks on $n-$groups, Zbornik radova Filozofskog fakulteta u Nišu, Serija Matematika 2(1988), 69–71.

CONTENTS